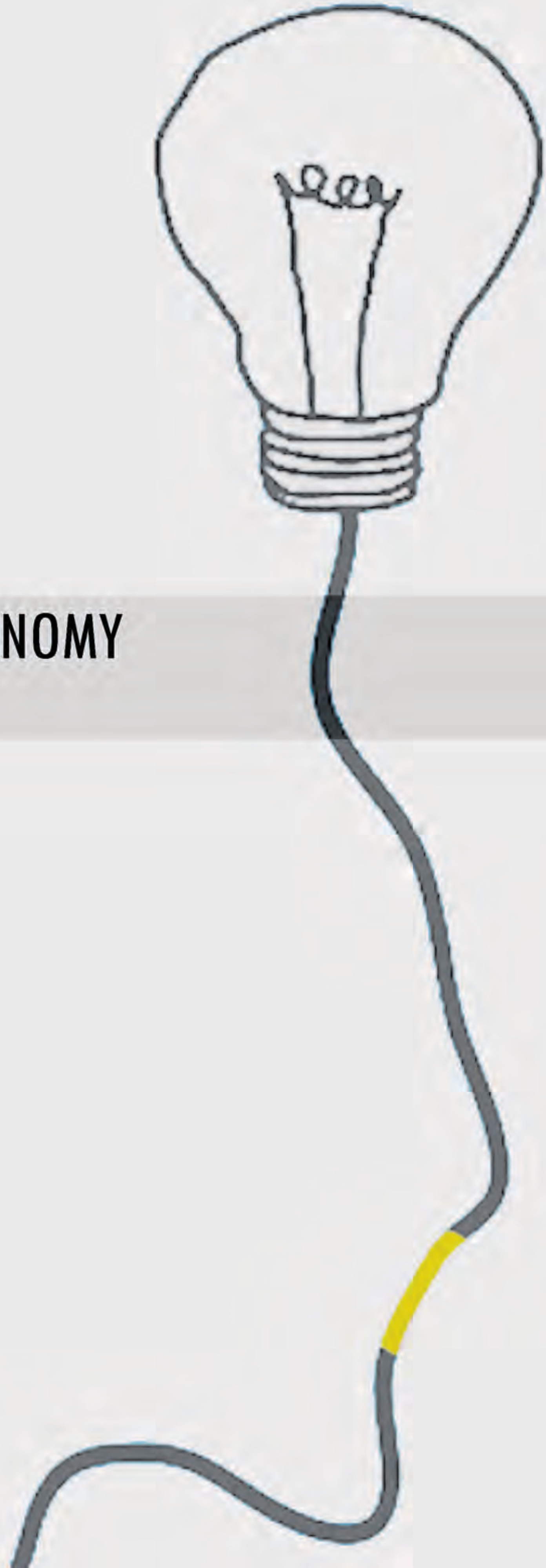
By: Ghena Hammour



INDE 301 | ENGINEERING ECONOMY

GHENA HAMMOUR

Chapter 1 - Foundations of Engineering Economy

Engineering Economy is at the heart of making decisions. These decisions involve the fundamental elements of cash flows of money, time and interest rates.

Three main Criterea used in making economic decisions:

1. If the input is fixed (fixed cost) → Maximize the output.
   Ex: Say, you are producing pens and you have a budget of \$100 to produce them → you try to maximize the number of pens produced (output)
2. If the output is fixed → Minimize the input.
   Ex: say you want to paint a room, You do that while using the least/minimal amount of paint
3. If neither the output nor the input are fixed then we take → maximum [output - input].

Essential keywords:

- Investment
- Cost.
- Useful life of an investment/project.
- Minimum Attractive Rate of Return (MARR) / Discount Rate / Hurdle rate.
- Salvage value.

- Investment:

Using capitals of money or funds to generate revenues (benefits) or provide a service.
Ex: Say you own a company, an investment might be when you buy computers for your employees, or when you provide them with refreshments. Note that you might not be able to measure that "money" wise.

- Cost:

All possible expenditures.
Ex: Initial cost, operating cost, maintenance cost, material cost, insurance cost...

- Useful life of an investment or project:

Duration time from the start till the end of an investment or project.

- Minimum Attractive Rate of Return (MARR) / discount rate.

Lowest rate accepted on the returns of a project or investment.
It is also the rate by which we discount the cash flow of the investment.
Ex: Say you have some amount of money deposited in the bank that gives you 5% on them. You get a project to invest this money with a rate of 4%. You won't agree because you got a min. return rate of 5%.

Salvage Value:

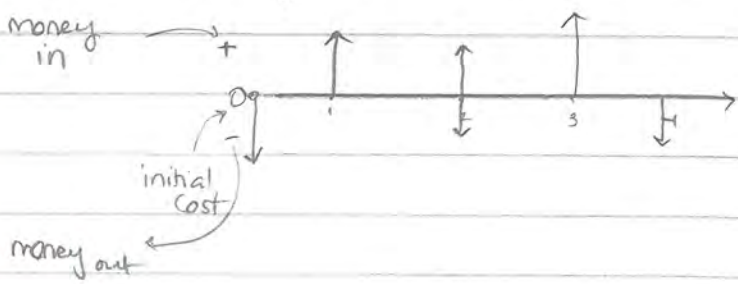
The value we get from an asset after a specified period of usage.

Ex: say you got a new car for 20,000 after some years you buy it for 12,000 ← salvage value.

Definitions:

Cash Flow:

A cash flow is a time graph or diagram that keeps track of all cash transactions within "n" periods or intervals of equal time. (most commonly n=1 year)



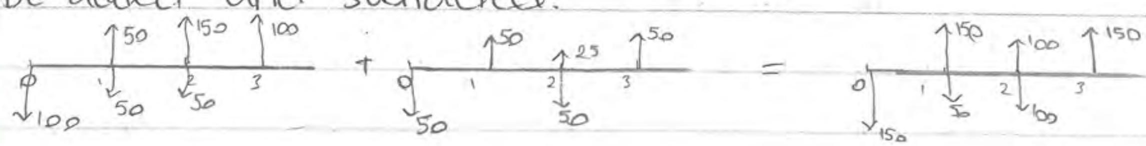
∴ we usually use the "end of period convention";

Remark 1-

In constructing cash flows, we adopt end of period convention that is, all transactions within a period are reported at the end of the period.

Remark 2-

Cash flows with the same number of equal periods can be added and subtracted.



- Interest Rate per period or rate of return
(there is a duality btw them, depending from which prespective we are looking: are you the bank or the customer)

Interest rate per period = $\frac{\text{amount of debt} - \text{original debt}}{\text{original debt}}$

Example:
You owe the bank \$100 at the end of 1 year for a loan of \$1000.
What is the interest rate per year charged? by the bank?

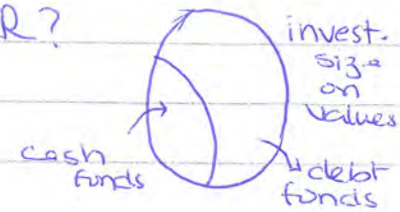
Interest rate per period = $\left[\frac{1100 - 1000}{1000}\right] \times 100 = 10\%$

Minimal Attractive rate of return;
- There is no "golden rule" for that because many things can't be calculated but are rather estimated.
- It is the lowest rate accepted on the return of an investment.
- It is higher than the rate offered by the commercial banks.
- It is actually determined by the financial analysts of the Company.
- It includes a risk factor.
- It may vary from a project to another.

MARR

critical MARR can be calculated as a weighted avg.

How to Calculate the critical MARR?



Let γ be the proportion of cash function.

Let i_c be the rate of return per/period on the cash funds.

Let i_b be the interest rate/period charge on the debt function.

critical MARR = $\gamma i_c + (1-\gamma) i_b$

Example.

For a given investment of \$15,000, the available cash is \$5000 gaining interest at 5% a year. The \$10,000 are to be borrowed at 8% a year. What is the critical MARR?

critical MARR = $\gamma i_c + (1-\gamma) i_b$

$\gamma = \frac{\$5,000}{\$15,000} = \frac{1}{3}$; proportion of cash function

$i_c = 5\%$; return rate per period

$i_b = 8\%$; interest rate/period charge on the debt function.

$$\text{Critical MARR} = \frac{1}{3}(5\%) + \left(1-\frac{1}{3}\right)(8\%)$$
$$= \frac{1}{3}(0.05) + \frac{2}{3}(0.08) = 0.07$$

critical MARR = 7%.

Time Value of money; Interest and Interest factors

Interest
- Simple
- Compound Interest
  - Discrete compounding
  - Continuous compounding

Simple Interest:

Interest is paid at the end of each period on the original sum of money only.

Ex: say you have \$1000 gaining an interest of 10%.

At the end of 1 year = \$1100

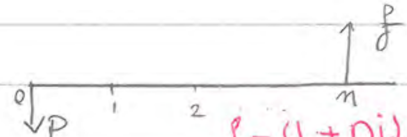
At the end of 2nd year = \$1200 (add 100 only to original)

Example.

i: interest rate per period

p: original sum of money

f: future value.



$f = (1 + ni)p$

Compound Interest:

Interest is paid on gained interest.

Discrete compounding:

Interest is paid discretely at the end of each period.

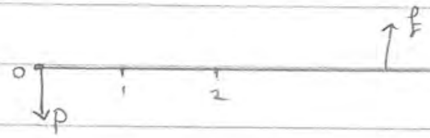
Discrete Compounding factors

- The case of single payment.

i: interest rate/period

p: original sum of money

f: future value or worth



End of period	Value	original amount subjected to interest + original
1	$(1+i)p$	$(1+i)p(i) + (1+i)p$
2	$(1+i)^2p$	$(1+i)p(1+i) = (1+i)^2p$
3	$(1+i)^3p$	$(1+i)^2p(i) + (1+i)^2p = (1+i)^3p$
$\vdots$	$\vdots$	$\vdots$
n	$(1+i)^n p$	

$$f = (1+i)^n p$$

The term $(1+i)^n$ is called the future worth factor of a single payment compounded at the rate i/period for "n" periods.
symbol : $(F \mid P, i, n)$ read as: f given p, i and n.

- What if we want to find the original sum P?

$$P = (1+i)^{-n} f$$

The term $(1+i)^{-n}$ is called present worth factor of a future value under discrete compounding at the rate i/period.
symbol : $(P \mid f, i, n)$ read as: P given f, i, n

A rule of thumb

The number of periods it takes a sum of money to double under compounding at the rate i/period is $\approx \frac{72}{i}$

Where did that come from?

$(1+i)^n p = 2p$

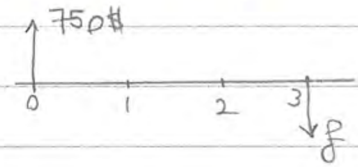
$n \ln(1+i) = \ln 2$

$n = \frac{\ln 2}{\ln(1+i)} \simeq \frac{\ln 2}{i - \underbrace{\frac{i^2}{2} + \frac{i^3}{3} + \frac{i^4}{4} \cdots}_{\text{neglect}}} \simeq \frac{\ln 2}{i} \simeq \frac{0.7}{i}$

Side Note: $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots \quad ; \quad -1 < x \leq 1$

Example:

A person borrows \$750 from a bank and agrees to repay the whole debt after three years. The interest charged on the loan is 8% compounded annually, how much should the person pay?

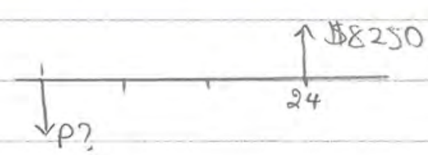


$f = (1+i)^n p$

$f = (1 + 8\%)^3 \times 750$

$f = 944.784 \simeq 945\$$

What sum of money now is equiv. to \$8250 two years from now, if the interest gained is 4% per month compounded every month.

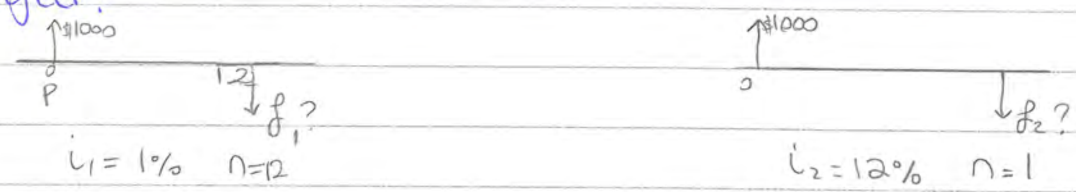


$p = (1+i)^{-n} f$

$p = (1 + 4\%)^{-24} \times 8250$

$p = \cancel{\$7052}\ \$3218.5$

A \$1000 can be borrowed at a rate of 1% a month compounded monthly for one year.
If the same amount can be borrowed for one year at 12% a year. How much could be saved in interest charged?



$f_1 = (1+i_1)^n P \quad ; \quad f_1 = 1000\,(F|P, 1\%, 12)$

$f_1 = (1+1\%)^{12} \times 1000$

$f_1 \simeq 1127\$$

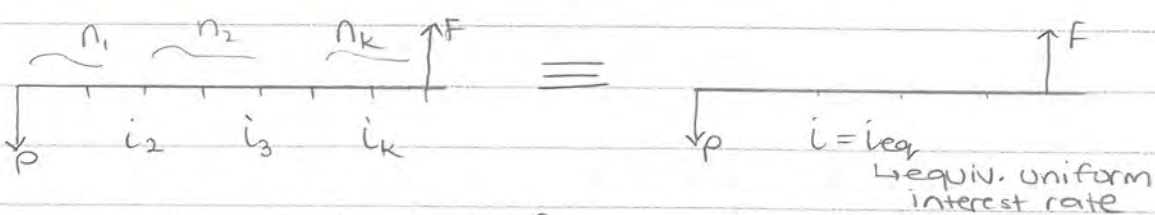
$f_2 = 1000\,(f_2|P, 12\%, 1)$

$f_2 = (1+12\%) \times 1000$

$f_2 = 1120\$$

→ amount saved is 7\$

Variable rates and the equivalent uniform rate.



$f = (1+i_1)^{n_1}(1+i_2)^{n_2} \cdots (1+i_k)^{n_k} P \equiv f = (1+i_{eq})^n P$

equate both sides, $f = f$

$(1+i_1)^{n_1}(1+i_2)^{n_2} \cdots (1+i_k)^{n_k} \cancel{P} = (1+i_{eq})^n \cancel{P}$

$(1+i_1)^{n_1}(1+i_2)^{n_2} \cdots (1+i_k)^{n_k} = (1+i_{eq})^n$

$i_{eq} = \sqrt[n]{\underbrace{(1+i_1)^{n_1}(1+i_2)^{n_2} \cdots (1+i_k)^{n_k}}_{\text{geometric mean}}} - 1$

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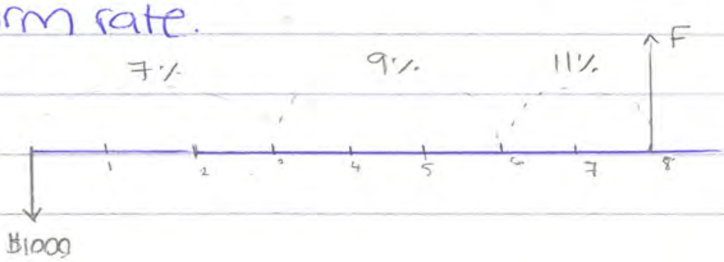
Example.
For the given cash flow, find "F" and the equivalent uniform rate.



Finding F

$$F = (1+i_1)^{n_1}(1+i_2)^{n_2}(1+i_3)^{n_3}\ldots(1+i_k)^{n_k}P$$

$$F = (1+7\%)^3(1+9\%)^3(1+11\%)^2 \times 1000$$

$$F = 1,954.68$$

$$F \simeq \$1955$$

Finding i_{eq} (equiv. uniform rate)

$$i_{eq} = \sqrt[n]{(1+i_1)^{n_1}(1+i_2)^{n_2}(1+i_3)^{n_3}\ldots(1+i_k)^{n_k}} - 1$$

$$i_{eq} = \sqrt[8]{(1+7\%)^3(1+9\%)^3(1+11\%)^2} - 1$$

$$i_{eq} = 8.7\ \%$$

this is also the avg.

$n_1, n_2 \ldots$ depends on how long was the interest rate for example 7%, $n=3$; 9%, $n=2$

When finding i_{eq}, n = total number of periods.

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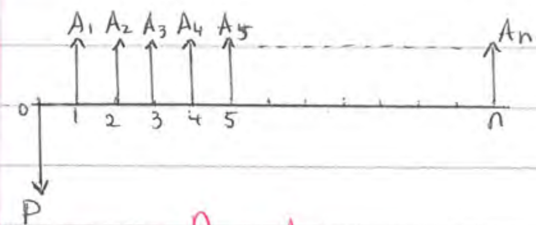
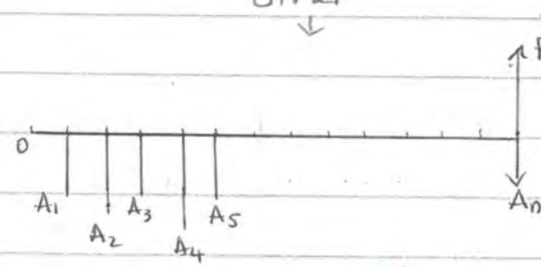
Discrete compounding factors for a series of payments or receipts.

(By convention; The series starts at period 1 and ends at period n).

General

*The payments here doesnt necescarily have to have a relation btw each other





$$P = \sum_{J=1}^{n} \frac{A_J}{(1+i)^J}$$

$$F = A_1(1+i)^{n-1} + A_2(1+i)^{n-2} + \dots + A_n$$

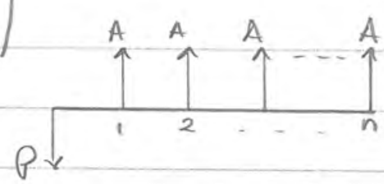
Special case 1

The case of a uniform series

$$A_1 = A_2 = A_3 = \dots = A_n = A$$

$$P = A \sum_{J=1}^{n} \left(\frac{1}{1+i}\right)^J = A\left(\frac{\left(\frac{1}{1+i}\right)^{n+1} - 1}{\frac{1}{1+i} - 1} - 1\right) = A\left[\frac{1-(1+i)^{n+1}}{-i(1+i)^n} - 1\right]$$

$$P = A\left(\frac{(1+i)^n(1+i-1) - 1}{i(1+i)^n}\right) = A\left(\frac{(1+i)^n - 1}{i(1+i)^n}\right)$$



$$* \quad P = A\left(\frac{(1+i)^n - 1}{i(1+i)^n}\right)$$

The term $\frac{(1+i)^n - 1}{i(1+i)^n}$ is called the present worth factor of a uniform series.

symbol: $(P|A, i, n)$ P given uniform series A at i percent for n periods

We can solve for A to find:

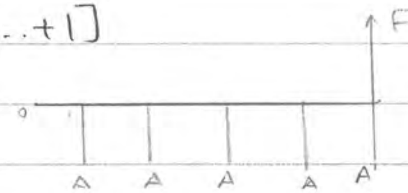
$$* \quad A = \left[\frac{i(1+i)^n}{(1+i)^n - 1} \right] P$$

The term $\frac{i(1+i)^n}{(1+i)^n - 1}$ is called the uniform series value factor given a present worth.

symbol: $(A|P, i, n)$.

Future worth

$$F = A[(1+i)^{n-1} + (1+i)^{n-2} + (1+i)^{n-3} + \dots + 1]$$



$$* \quad F = A\left[\frac{(1+i)^n - 1}{i} \right]$$

The term $\left(\frac{(1+i)^n - 1}{i} \right)$ is called the future worth factor of a uniform series.

symbol: $(F|A, i, n)$

We can solve for A to find:

$$* \quad A = \left[\frac{i}{(1+i)^n - 1} \right] F$$

The term $\left[\frac{i}{(1+i)^n - 1} \right]$ is called the uniform series factor value given a future worth.

symbol: $(A|F, i, n)$

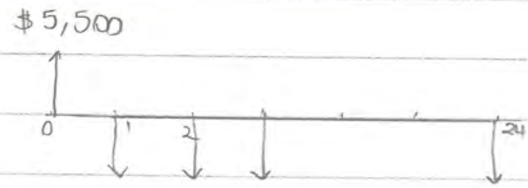
Example

You borrow \$5500 from the bank to be repaid in equal monthly installments starting next month for 2 years.
The interest charged is 9% a year compounded monthly.
Calculate the value of this monthly payment.

$n = 2$ years $= 24$ months

interest $= 9\%$ per year

interest per month $= \frac{9\%}{12} = 0.75\%$



(We are asked to find the monthly payment. This is a uniform series A given P)

$(A|P, i, n)$

$$A = \left[\frac{i(1+i)^n}{(1+i)^n - 1}\right]P = \left[\frac{0.75\%\,(1+0.75\%)^{24}}{(1+0.75\%)^{24} - 1}\right] \times 5{,}500$$

$A = \$\,251$

By: Ghena Hammour

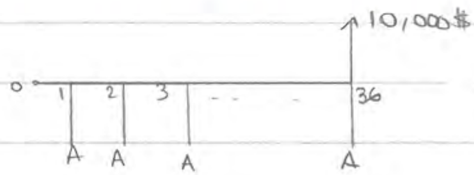
Example

You plan to save \$10,000 in 3 years by making equal monthly deposits in a savings account in a given bank starting next month. The interest gained is 6% a year compounded monthly. What is the value of this monthly deposit?

$n = 3$ years $= 36$ months

$i = 6\%$ per year

$i/\text{month} = \frac{6\%}{12} = 0.5\%$



We want to find the value of monthly deposit which is A, a uniform series.

$(A \mid F, i, n)$

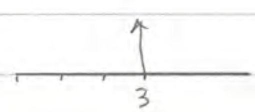
$$A = \left(\frac{i}{(1+i)^n - 1}\right) F$$

$$A = \left(\frac{0.5\%}{(1+0.5\%)^{36} - 1}\right) \times 10{,}000$$

$A = \$254$

Note

A series may not start at period 1, maybe at some other period, say for instance at 3.

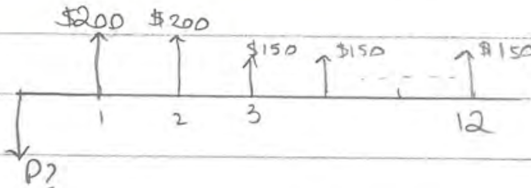


Now, if you find the present worth it is actually one period before which is at 2, so this isn't really the present worth, you must multiply by the future factor.

Example

imp * How much money should be deposited now to withdraw \$200 a month for the first two months then \$150 a month till the end of the year. The interest gained is 6% a year compounded monthly.

i permonth = 6%/12 = 0.5%



Now here we "break" it into two, the first two periods and the rest of the periods (the rest are now shifted and here you should multiply by the factor.

In general, For a uniform series (P|A, i, n)

$$P = A\left(\frac{(1+i)^n - 1}{i(1+i)^n}\right),$$

Note this is for shift difference!

$$\left(\frac{i(1+i)^n}{(1+i)^n - 1}\right)$$ this is the factor

$$P = A_1\left(\frac{(1+i)^n - 1}{i(1+i)^n}\right) + A_2\left(\frac{(1+i)^n - 1}{i(1+i)^n}\right) \times (P|F, i, n)$$

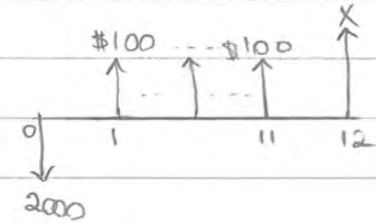
$$P = 200\left(\frac{(1+0.5\%)^2 - 1}{0.5\%\,(1+0.5\%)^2}\right) + 150\left(\frac{(1+0.5\%)^{10} - 1}{0.5\%\,(1+0.5\%)^{10}}\right)\left(\frac{0.5\%\,(1+0.5\%)^2}{(1+0.5\%)^2 - 1}\right)$$

P = \$ 1842

Example

You deposit \$2000 at the rate of 9% a year compounded monthly.
You plan to make 11 monthly withdrawls of \$100 then on the 12th month you draw all what is left.
How much will be you final withdrawl?

interest per month = $\frac{9\%}{12} = 0.75\%$

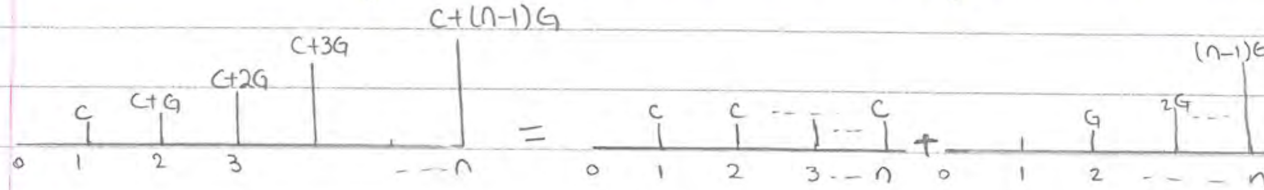


we are given the present worth
$P = \$2000$
We take it as a sum of two things.
$P = (P|A, i, n) + (P|F, i, n)$

$2000 = 100\,(P|A, 0.75\%, 11) + x\,(P|F, 0.75\%, 12)$
$2000 = 100\,(10.5207) + x\,(0.9142)$
solving for x
$X = 1036.89 \simeq 1037\,\$$

Special Case 2

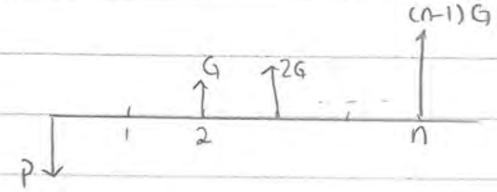
Arithmetic gradient series with a gradient "G".



For the Arithmetic gradient series we compute the following:

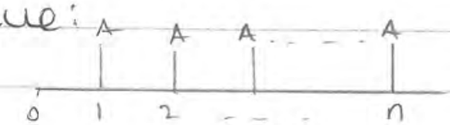
Present worth:

* $P = G\left[\frac{(1+i)^n - in - 1}{i^2(1+i)^n}\right]$ →



The equivalent uniform series value:

* $A = G\left[\frac{1}{i} - \frac{n}{(1+i)^n - 1}\right]$



The term $\left[\frac{(1+i)^n - in - 1}{i^2(1+i)^n}\right]$ is called the present worth factor of an arithmetic gradient "G"

symbol: $(P|G, i, n)$

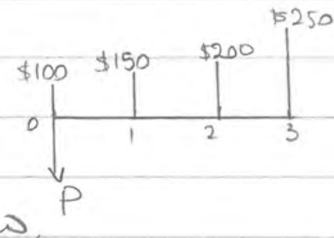
The term $\left(\frac{1}{i} - \frac{n}{(1+i)^n - 1}\right)$ is called the equivalent uniform series factor for an arithmetic sequence/series.

symbol: $(A|G, i, n)$

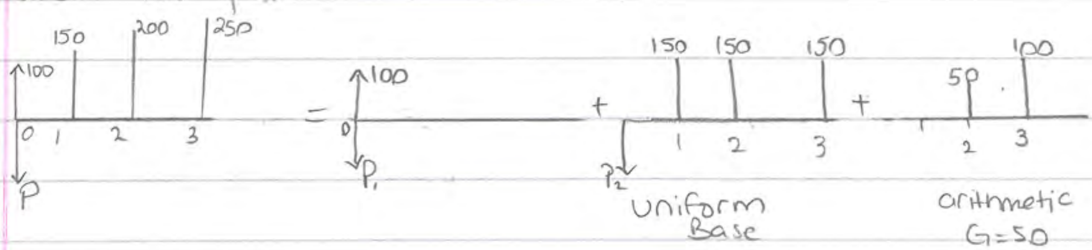
Example

Find the present worth of the given cash flow.

The arithmetic series starts at period 2, so we will split the cash flow.



$i = 10\%$



$$P = P_1 + P_2 + P_3$$

$$P = 100 + 150(P|A, 10\%, 3) + 50(P|G, 10\%, 3)$$

$$P = 100 + A\left(\frac{(1+i)^n - 1}{i(1+i)^n}\right) + G\left(\frac{(1+i)^n - in - 1}{i^2(1+i)^n}\right)$$

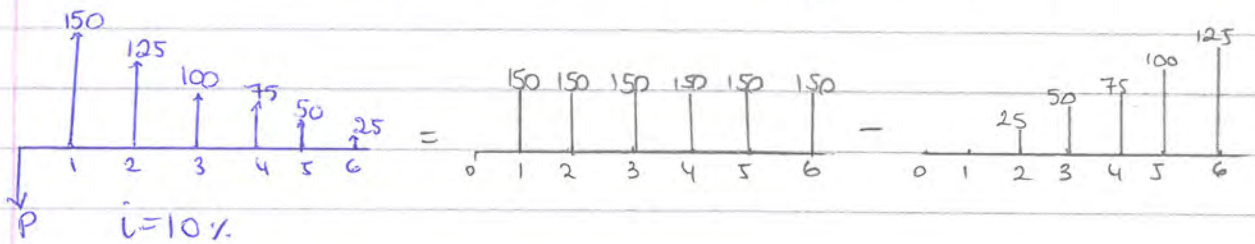
$$P = 100 + 150\left(\frac{(1+10\%)^3 - 1}{10\%(1+10\%)^3}\right) + 50\left(\frac{(1+10\%)^3 - 10\% \times 3 - 1}{(10\%)^2(1+10\%)^3}\right)$$

$$P = 100 + 373.0277 + 116.45379$$

$$P \simeq \$590$$

Example

Find the present worth of the given cash flow.



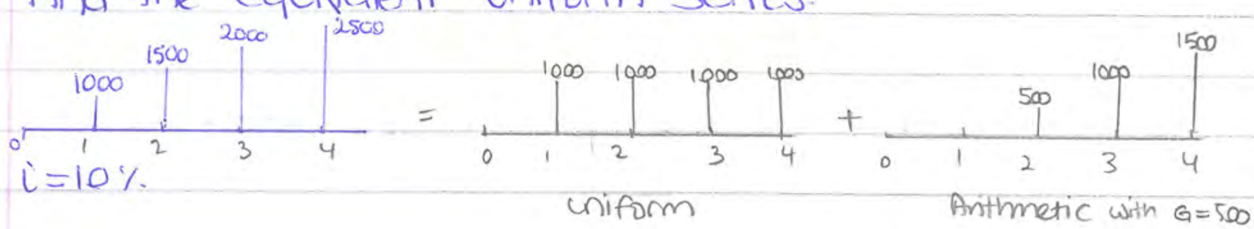
$$P = (P|A, i, n) - (P|G, i, n)$$

$$P = (P|150, 10\%, 6) - (P|25, 10\%, 6)$$

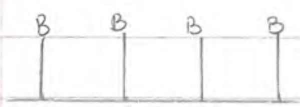
$$P = 150\left(\frac{(1+10\%)^6 - 1}{10\%(1+10\%)^6}\right) - 25\left(\frac{(1+10\%)^6 - 6(10\%) - 1}{(10\%)^2(1+10\%)^6}\right)$$

$$P = \$411.18$$

Find the equivalent uniform series.



The equivalent uniform series.



$$= 1000 + (A|G, i, n)$$

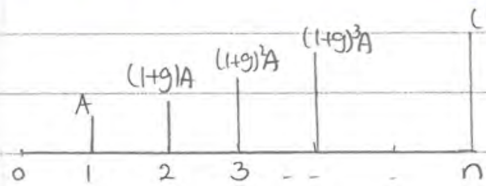
$$= 1000 + G\left(\frac{1}{i} - \frac{n}{(1+i)^n - 1}\right)$$

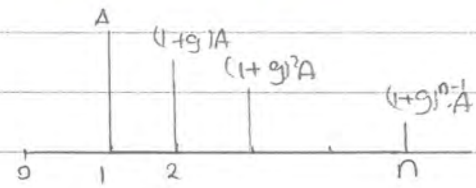
$$= 1000 + 500\left(\frac{1}{10\%} - \frac{4}{(1+10\%)^4 - 1}\right)$$

$$= \$1690.5$$

Special Case 3

Geometric series of payments or receipts.





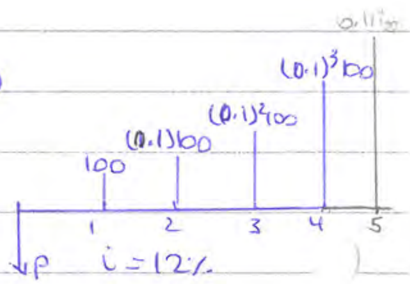
* Increasing sequence with g positive
* Decreasing sequence with g negative

For such sequences the present worth is:

$$P = \begin{cases} A\left[\dfrac{1-\left(\frac{1+g}{1+i}\right)^n}{i-g}\right], & i \neq g \\ \dfrac{nA}{1+i} & i = g \end{cases}$$

Example.

Find P for the given cash flow

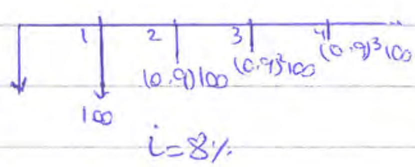


This is a geometric sequence

$i \neq g$; $i = 12\%$ $g = 0.1$

$$P = 100\left[\frac{1-\left(\frac{1+0.1}{1+12\%}\right)^5}{12\% - 0.1}\right] = \$430$$

Find P for the given cash flow



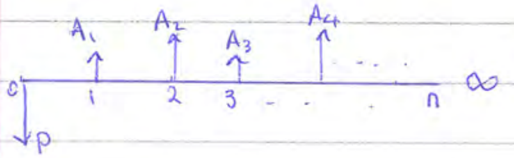
This is a geometric sequence

$i \neq g$; $i = 8\%$ $g = -0.1$

$$P = 100\left[\frac{1-\left(\frac{1-0.1}{1+8\%}\right)^4}{8\% - (-0.1)}\right] = \$287$$

Infinite Series of Payments or receipts

Consider the infinite series



$$\sum_{i=j}^{\infty} (1+i)^{j} A_j = \infty$$

this is a divergent series.

$$P = \sum_{j=1}^{\infty} \frac{A_j}{(1+i)^j}$$

Special Case:

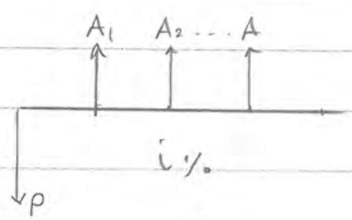
When the series can be handled using geometric series.

$$\sum_{J=j_0}^{\infty} x^J = \frac{x^{j_0}}{1-x}, \quad |x|<1$$

Special Case 1_

When $A_1 = A_2 = A_3 = \dots = A$

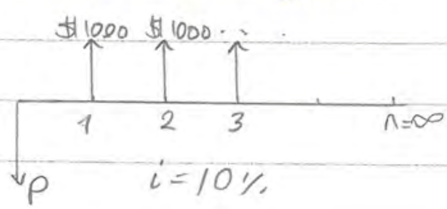
$$P = A\sum_{j=1}^{\infty}\left(\frac{1}{1+i}\right)^j = A\left[\frac{1/(1+i)}{1-1/(1+i)}\right] = \frac{A}{i}$$



$$P = \frac{A}{i}$$

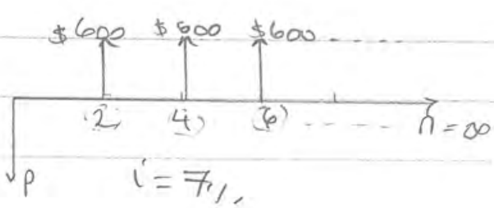
Example. Find the present worth "P" for the given Cash Flow.

(1) $P = \frac{A}{i} = \frac{1000}{10\%} = \$10,000$



trick (2) $P = A\sum_{J=1}^{\infty}\left(\frac{1}{1+i}\right)^J = A\left(\frac{1/(1+i)}{1-1/(1+i)}\right)$

$$P = 600\sum_{j=1}^{\infty}\frac{1}{(1.07)^{2j}} = 600\sum_{j=1}^{\infty}\left[\left(\frac{1}{1.07}\right)^2\right]^J$$

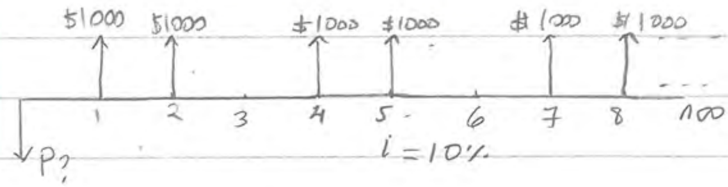


$$P = 600\cdot\left[\frac{1/(1.07)^2}{1-1/(1.07)^2}\right] = \$4141$$

Take Home.

Find the present worth for the given Cash flows.

1



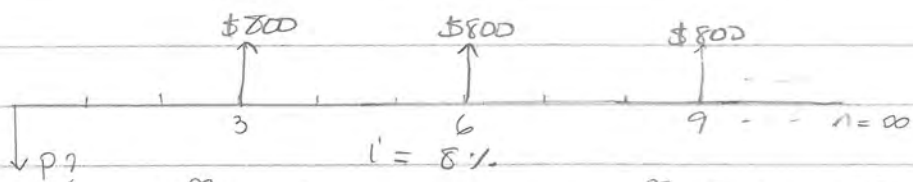
$$P_1 = \frac{A}{i} = \frac{1000}{10\%} = \$10{,}000$$

$$P_2 = 1000\left(\frac{1/(1+10\%)^3}{1 - 1/(1+10\%)^3}\right) = \$3021.14$$

$P = 10{,}000 - 3021.14$

$P = 6978.85\$$

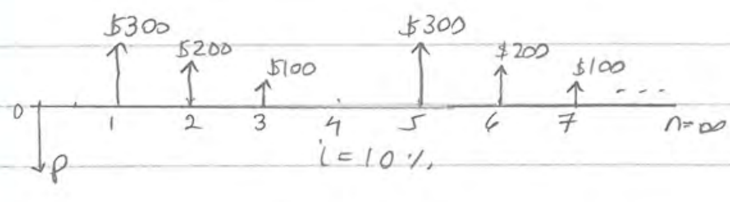
2



$$P = 800 \sum_{J=1}^{\infty} \frac{1}{(1+8\%)^{3J}} = 800 \sum_{j=1}^{\infty} \left[\frac{1}{(1+8\%)^3}\right]^J$$

$$P = 800\left[\frac{1/(1+8\%)^3}{1 - 1/(1+8\%)^3}\right] = \$3080.33$$

3



$$P_1 = \frac{A}{i} = \frac{100}{10\%} = \$1000$$

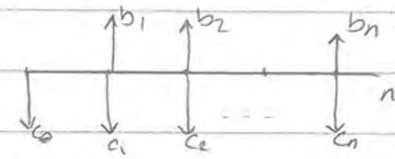
$$P_2 = 100\left(\frac{1/(1+10\%)^4}{1 - 1/(1+10\%)^4}\right) = 215.470$$

$1000 - 215.470 = 784.529$

?

Balancing a cash flow under a given Interest rate.

Consider an arbitrary cash flow



We define:

- Present Worth of the upper flow =
  sum of the present worth for all b_j
- Present Worth of the lower flow =
  sum of the present worth for all c_j.
- future Worth of the upper flow =
  Sum of the future worth for all b_j
- future Worth of the lower flow =
  Sum of the future worth for all c_j.

We say that the cash flow is balanced under the interest rate "i" if:

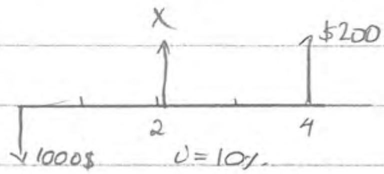
present Worth of upper flow = Present Worth of lower flow

or

future Worth of upper Flow = future Worth of lower flow.

Example. For the balanced Cash flow. Find X.

$P = P_1 + P_2$



$1000 = X(P|F, 10\%, 2) + 200(P|F, 10\%, 4)$

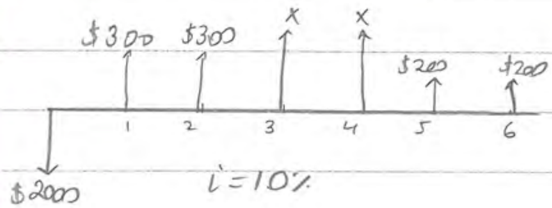
$1000 = X(0.8264) + 200(0.6830)$

Solving for X

$X = \$1044.77 \simeq \$1045$

Take home

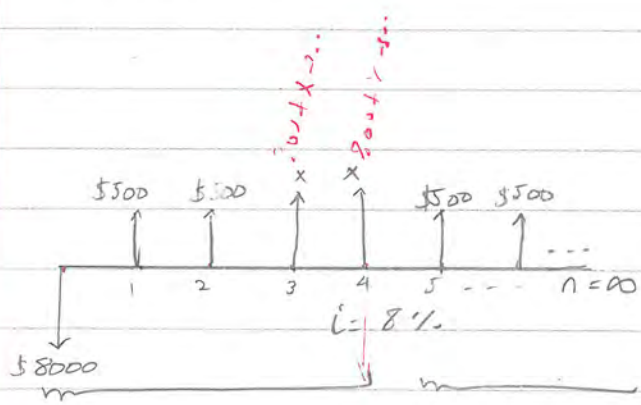
Find x for the balanced cash flows:



$$2000 = 300\,(P/A, 10\%, 2) + x\,(P/A, 10\%, 2)(P/F, 10\%, 2)$$
$$+ 200\,(P/A, 10\%, 2)(P/F, 10\%, 4)$$

$$2000 = 300\,(1.7355) + x\,(1.7355)(0.8264) + 200\,(1.7355)(0.6830)$$

$x = 866.17\$$



$$8000 = 500\,(P/A, 8\%, 2) + x\,(P/A, 8\%, 2)(P/F, 8\%, 2)$$
$$+ \frac{500}{8\%} \cdot (P/F, 8\%, 4)$$

$$8000 = 500\,(1.7833) + x\,(1.7833)(0.8573) + \frac{500}{8\%}\,(0.7350)$$

$x = 1644.79\$$

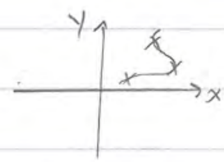
you still need to shift to the present at 0.

By: Ghena Hammour

Evaluation of the interest Factors:

- use the table
  - Find in table
  - don't find in table
    - use interpolation
- substitute in th formula

Side Note!



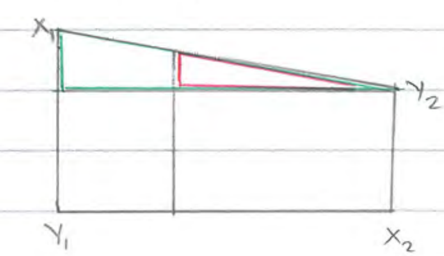
Interpolator: any function that passes through all the data points.

However, the problem is that you can have ∞ many intepolators.

The best/widely used are the polynomials.

Linear spline interpolator, for our application we need interpolation of a line.

Similarity in triangles:



Example. Evaluate the $(A|P, 8.35\%, 10)$ factor.

One way is to use the formula:

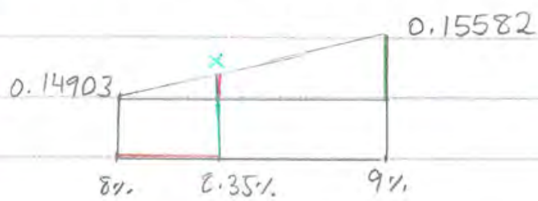
$$A = P \cdot \left(\frac{i(1+i)^n}{(1+i)^n - 1}\right) = \frac{8.35\%\,(1+8.35\%)^{10}}{(1+8.35\%)^{10} - 1} = 0.15139$$

We can also calculate it using Interpolation and the tables:

We look at the table for $i = 8.35\%$ we can't find!

- $i = 8\%$, Uniform series Payments, (Capital Recovery)
  $n = 10$ $(A/P) = 0.14903$
- $i = 9\%$, $(A/P) = 0.15582$

Now $i = 8.35\%$ lies between 8% and 9% and we can use interpolation.



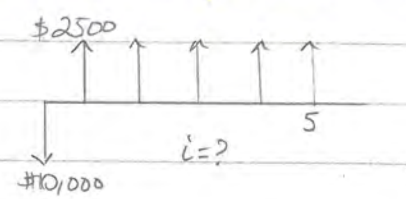
$$\frac{x - 0.14903}{0.15582 - 0.14903} = \frac{8.35\% - 8\%}{1\%}$$

$$x = \frac{0.0035\,(6.79\times10^{-3})}{0.01} + 0.14903$$

$$x = 0.1514$$

Example. For the given cash flow find the balancing interest rate.

$P = 10{,}000\$$ $n = 5$ $A = 2{,}500$



$$(P|A, i, n) = A \cdot \left(\frac{(1+i)^n - 1}{i(1+i)^n}\right)$$

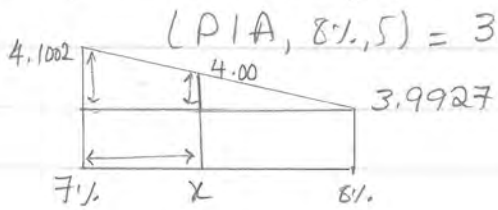
$$10{,}000 = 2{,}500 \times \text{''}$$

$$\Rightarrow (P|A, i, n) = \frac{10{,}000}{2500} = 4$$

if you want to solve it you'll need numerical methods $\Rightarrow$ Use table.

for $n = 5$ $(P|2500, i, 5) = 4$, if we can't find 4 then find close

$n = 5$, $(P|A, 7\%, 5) = 4.1002$
$(P|A, 8\%, 5) = 3.9927$



Using Similarity of Δ's

$$\frac{4 - 3.9927}{4.1002 - 3.9927} = \frac{1\% - x}{1\%}$$

$$x = 0.00933$$

$$\Rightarrow i = 7\% + 0.932 \simeq 7.933\%$$

By: Ghena Hammour

Nominal Rates, Effective Rates and Continuous Compounding

Nominal Rates:

An interest rate stated per period is called the nominal rate per period and denoted by "r".

- If the original period (nominal period) is subdivided into "m" equal subperiods then the rate per subperiod is: $\frac{r}{m}$

- If the original period is multiplied by a positive integer "k" then the rate per "k" periods is $k \times r$

Example. If the nominal rate is 12% per year then: it is what

1 year → 12 month   12% → 12 month
x → 1 month

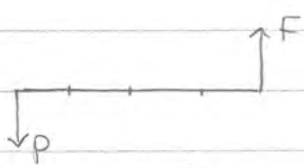
r/m per month: $(12\%/12) \times 100 = 1\%$

r/m per quarter of a year: $(12\%/4) \times 100 = 3\%$

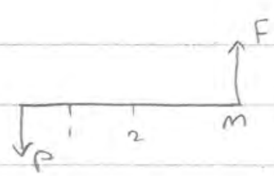
r/m per half a year: $(12\%/2) \times 100 = 6\%$

$k \times r$ per 2 years: $12\% \times 2 = 24\%$

Side Note (After effective rate)



instead of the subperiods;



$i_a = (1+r)^k - 1$ $i_a = (1+\frac{r}{m})^m - 1$

Effective interest rate:

The effective rate of compounding by the sub period.

r: nominal rate / period

The period is divided into "m" equal subperiods and the compounding is done by the subperiod.

$i = i_a$ (effec. rate / orignal period)

$F = (1+i_a)P$

$i = \frac{r}{m}$

$F = (1+\frac{r}{m})^m P$

$F = F$.

$(1+i_a)\cancel{P} = (1+\frac{r}{m})^m \cancel{P}$

$$i_a = \left(1+\frac{r}{m}\right)^m - 1$$

Example. Compute the effective rate per year for a nominal rate of 12% a year compounded:

- quarterly: $i_a = \left(1+\frac{12\%}{4}\right)^4 - 1 = 12.55\%$
- monthly: $i_a = \left(1+\frac{12\%}{12}\right)^{12} - 1 = 12.68\%$
- weekly: $i_a = \left(1+\frac{12\%}{52}\right)^{52} - 1 = 12.73\%$
- daily: $i_a = \left(1+\frac{12\%}{365}\right)^{365} - 1 = 12.75\%$

There is no much gain when making more subdivisions because there is some limit that is reached.

By: Ghena Hammour

Example. Which has the highest effective annual rate?

- 9% a year compounded quarterly?

  $i_a = \left(1 + \frac{9\%}{4}\right)^4 - 1 = 9.3\%$

- 2% a quarter compounded quarterly?

  $i_a = \left(1 + \frac{2\%}{1}\right)^4 - 1 = 8.24\%$

Like the note on previous pg. $i_a = (1+r)^k - 1$

- 8.8% a year compounded monthly?

  $i_a = \left(1 + \frac{8.8\%}{12}\right)^{12} - 1 = 9.16\%$

A company plans to place money in a new venture capital fund that returns 18% a year compounded daily. What is the effectiv rate

- Per year:

  $i_a = \left(1 + \frac{18\%}{365}\right)^{365} - 1 = 19.7\%$

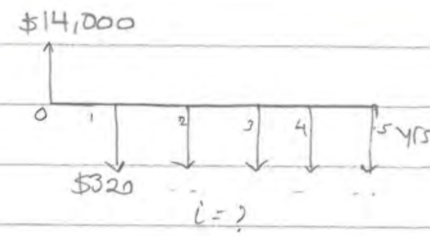
- per month:

  $i_a = \left(1 + \frac{18\%}{365}\right)^{30} - 1 = 1.49\%$

Byblos bank is offering the personal loan of $14,000 at a payment of $320 per month for 5yrs.

- What is the monthly interest rate?

$$(A \mid P, i, n) = P\left(\frac{i(1+i)^n}{(1+i)^n - 1}\right)$$



$P = 14{,}000$

$A = 320$ per month

$n = 5 \times 30 = 60$ months

$$320 = 14{,}000 \cdot \left(\frac{i(1+i)^{60}}{(1+i)^{60} - 1}\right)$$

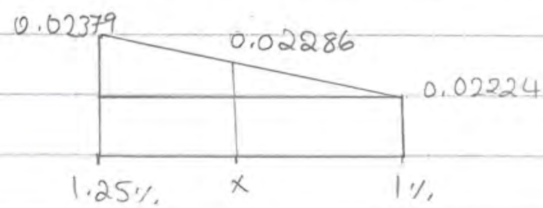
$i = 1.1\%$

we can also use the table;

$n = 60$ $\quad (A \mid P, i, 60) = \frac{320}{14{,}000} = 0.022857$ ← look at sth. near.

$(A \mid P, 1.25\%, 60) = 0.02379$

$(A \mid P, 1\%, 60) = 0.02224$



$$\frac{0.02286 - 0.02224}{0.02379 - 0.02224} = \frac{x}{1.25\% - 1\%}$$

$x = 0.001$

$i = (0.01 + 0.001) \times 100 = 1.1\%$

- What is the nominal annual rate?

$i = (1.1\%) \times 12 = 13.2\%$

- What is the effective annual rate?

$i_a = (1 + 1.1\%)^{12} - 1 = 14\%$

Continuous Compounding

r: nominal rate per period

period is subdivided into equal "m" subperiods.

Let

$\Delta t = \frac{1}{m}$ be the duration of the subperiod.

Continous compounding is achieved by letting $\Delta t \to 0$ or equivalently $m \to \infty$

The effective rate / period under continuous compounding is:

$$i_a = \lim_{m\to\infty} \left(\left(1+\frac{r}{m}\right)^m - 1\right) = e^r - 1$$

The interest is r/period and the compounding is continuous.

$$\lim_{m\to\infty} f = p.\left[\lim_{m\to\infty}\left(1+\frac{r}{m}\right)^m\right] = e^r.p$$

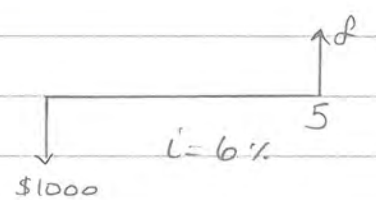
Consider continuous Compounding for "k" periods:

$f = e^{kr}.p$

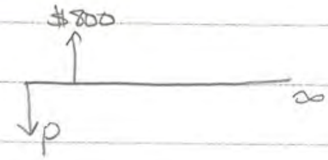
$p = f.e^{-kr}$

Example. Find the future worth under continuous compounding.

$$f = e^{5(6\%)}.1000 = \$1,350$$



Example. Find the present worth of the given cash flow at $i = 8\%$ a year compounded continuously.



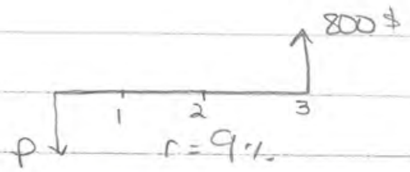
usually for an infinite series of payments (Discrete):

$$P = \frac{A}{i}$$

here it is compounded continuousily so $i_a = e^r - 1$

$$P = \frac{A}{e^r - 1} = \frac{800}{e^{8\%} - 1} \simeq \$9605$$

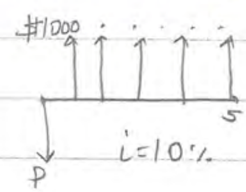
Example. Calculate the present worth under continuous compounding.



$$P = f \cdot e^{-kr}$$

$$P = 800 \cdot e^{-(3)(9\%)} = \$611$$

Example. Find the present worth of the uniform series of receipts at a rate of 10% and continuous compounding.



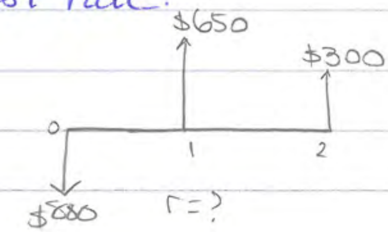
Usually we find $(P|A, i, n)$

$$P = A \cdot \left(\frac{(1+i)^n - 1}{(1+i)^n i} \right)$$

for continuous compounding $i = e^r - 1$

$$P = 1000 \left(\frac{(1 + e^{10\%} - 1)^5 - 1}{(1 + e^{10\%} - 1)^5 (e^{10\%} - 1)} \right) = \$3741.23$$

Example - For the balanced Cash flow under continuous compounding, what is the balancing interest rate.



$p = e^{-kr} \cdot f$

$800 = e^{-r} \cdot 650 + e^{-2r} \cdot 300$

let $x = e^{-r}$

$800 = 650x + 300x^2$

solve for x

$x = 0.8763$

$e^{-r} = 0.8763$

$r = -\ln(0.8763) \times 100 \simeq 13.21\%$.

Main Economic Analysis Methods.

- Present/future worth
- Annual worth
- Rate of return
- Benefit-Cost ratio.

! All methods will lead to the same decision.

Some common features used in all:

- Costs:

Include purchasing cost, operation & maintainence cost, insurance cost and all possible cost components.

- Benefits:

All revenues generated from a project or investment. (Sometimes a benefit like a salvage value in a pure service investment is a reduction in cost).

- Useful life:

Duration time of a project or investment.

- Salvage Value:

Value you get from an asset after usage.

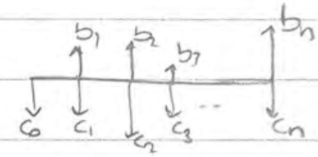
- Discount rate:/ MARR

An interest rate at which the cash flow of an investment is discounted

Types of Investments:

- Revenue Investments:

The revenue generated / or benefits depend on the choice of the investment.

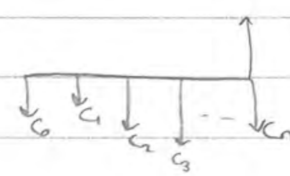


Example: investing in wind energy, in solar power, shopping malls, investing in a new product.

Given many competing revenue investments, if all are non attractive then we dont have to select any of them. (Do nothing option)
Main purpose: Generate profit.

- Service Investments:

The benefits are mainly difficult to assess financially.



Given many competing (mutually exclusive) service investments then exactly one must be selected.
Main purpose: provide the service.

* Mutually exclusive Investments:
we can select one at most.

* Independent investments:
We can select as many as our budget allows.

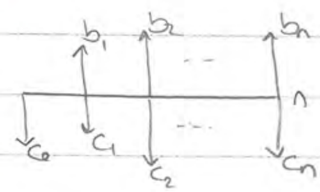
The present worth Economic Analysis Method.

Is an economic analysis technique that assesses the economic worth of a single revenue investment or selects the most attractive among many competing mutually exclusive revenue or service investments.

How to apply the method?

1- Case of single revenue investment:

At an interest rate = MARR we compute.



- PWB: present worth of all benefits
- PWC: present worth of all costs.

$$PW = PWB - PWC$$

if:

$$\begin{cases} PW \geq 0 & \text{investment is attractive} \\ PW < 0 & \text{investment is not attractive} \end{cases}$$

Example:

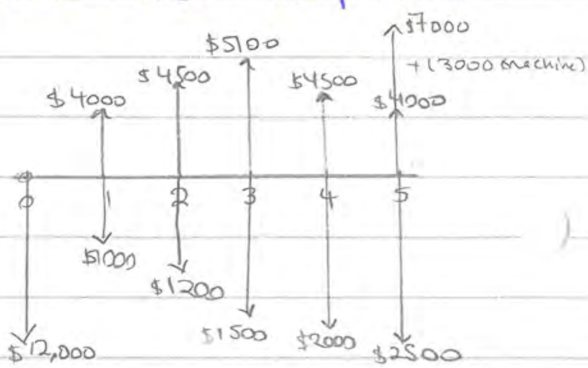
The production manger of a company is considering the purchase of a new machine that will be useful for 5 yrs

The purchase and installation cost is $12,000.

The costs and benefits for the next 5yrs are:

Year	Costs	Benefits
1	$1000	$4000
2	$1200	$4500
3	$ 1500	$ 5100
4	$ 2000	$4500
5	$ 2500	$ 4000

The MARR = 15% a year and the machine will be salvaged for $3000 at the end of the 5yrs. Is this investment attractive? Use the present worth method.



PW = PWB + PWC

PWB we calculate from:

$(P|F, i, n) = (1+i)^{-n} F$

at each n

PWC we calculate also:

$(P|F, i, n) = (1+i)^{-n} F$

PWB = 4000(P|F, 15%, 1) + 4500 (P|F, 15%, 2) + 5100 (P|F, 15% 3)
+ 4500(P|F, 15%, 4) + 7000 (P|F, 15%, 5)

PWB = $ 18, 607.

PWC = 12,000 + 1000(P|F, 15%, 1) + 1200 (P|F, 15% 2) + 1500 (P|F, 15%, 3)
+ 2000(P|F, 15%, 4) + 2500 (P|F, 15%, 5)

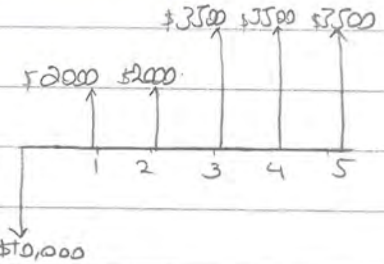
PWC = 17,996

PW = PWB − PWC > 0

∴ Attractive!

Example. An investment of \$10,000 returns annual benefits of \$2000 for the first two years and \$3500 for the last 3 years.
Is this investment attracting. The MARR = 10% a year? Use present worth analysis.

$PW = PWB - PWC$



$(P|A, i, n) = A \cdot \left(\frac{(1+i)^n - 1}{i(1+i)^n}\right)$

$PWB = 2000\,(P|A, 10\%, 2) + 3500\,(P|A, 10\%, 3) \times (P|F, 10\%, 2)$

for the shift we need to multiply by this factor

$= \left(\frac{i(1+i)^n}{(1+i)^n - 1}\right)$

Using the table

$PWB = 2000\,(P/A, 10\%, 2) + 3500\,(P/A, 10\%, 3) \times (P/F, 10\%, 2)$

$= 2000\,(1.7355) + 3500\,(2.4869)(0.8264)$

$PWB = 10,664$ in class 12,174 \$

$PWC = 10,000$

$PW = 10,664 - 10,000 \geq 0$ $\therefore$ Investment is attractive.

Example An investment of \$10,000 is good for 5 years. The expected costs are:

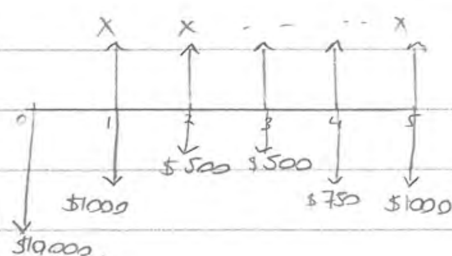
Year	Cost
1	1000
2	500
3	500
4	750
5	1000

If the MARR is 10% a year, what is the value of the minimum annual uniform benefit so that the investment is attractive? Use present worth analysis.

Let X = Uniform annual benefit

$PW = PWB - PWC$

$(P|F, i, n) = (1+i)^{-n} F$



$PWC = 10{,}000 + 1000(P|F, 10\%, 1)$
$\quad + 500\,[(P|F, 10\%, 2) + (P|F, 10\%, 3)]$
$\quad + 750(P|F, 10\%, 4) + 1000(P|F, 10\%, 5)$

$PWC = 12{,}831$

$PWB = (P|A, 10\%, 5)$

$$(P|A, 10\%, 5) = X \cdot \left(\frac{(1+i)^n - 1}{i(1+i)^n}\right) = X\left(\frac{(1.1)^5 - 1}{0.1\,(1.1)^5}\right) = 3.7908X$$

$PWB = 3.7908\,X$

$PW = 3.7908\,X - 12{,}831 \geq 0$

$$\Rightarrow X \geq \frac{12{,}831}{3.7908} = 3384$$

$X \geq \$3384.77$

2- Case of many mutually exclusive revenue investments.

- Select a common analysis period "T". In case all investments have the same useful life then "T" is simple the common value of the useful life.

- For the "J^{th}" investment we compute PW_J.

- Select the investment with the highest present worth postive is the most attractive. If $PW_J < 0 \ \forall_J$, then we don't have to select any - Do nothing option.

Example. A manager is concidering the two exclusive alternatives of buying one of two machine that cost \$1000 each. The estimated annual benefits are given in the table: The company's MARR = 10% a year which of the two machines if any is better buy? Use presen worth.

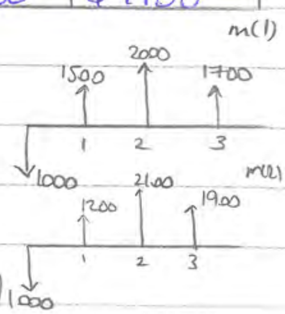
Year	machine (1)	machine (2)
1	\$1500	\$1200
2	\$2000	\$2100
3	\$1700	\$1900

$PWB_1 = 1500(P/F, 10\%, 1) + 2000(P/F, 10\%, 2) + 1700(P/F, 10\%, 3) =$

$PWB_1 = 1500(0.9091) + 2000(0.8264) + 1700(0.7513)$

$PWB_1 = 4293.6\$$

$PW_1 = 4294 - 1000 = \boxed{3294\ \$}$



$PWB_2 = 1200(P/F; 10\% 1) + 2100(P/F, 10\%, 2) + 1900(P/F\ 10\%\ 3)$

$PWB_2 = 1200(0.9091) + 2100(0.8264) + 1900(0.7513) = \$4254$

$PW_2 = 4254 - 1000 = \boxed{3254\$}$

Choose Machine 1

3. Case of many exclusive service investments

- Select a common analysis period "T."
- For the "Jth" investment we compute:

PWC_J = present worth of all costs − present worth of the salvage value

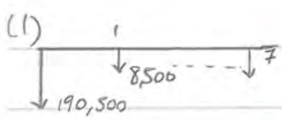
- The investment with the minimum cost PWC is the most attractive.

! The main difference btw service and revenue investment is that you don't have the do nothing option. You should select 1.

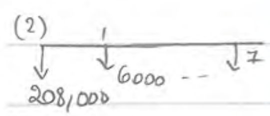
Example. A quality control Engineer in a factory that manufactures canned food is considering the automization of the quality control process.
A new investment is to be installed to randomly select and test samples of the products. The selection process rested on 3 investments.

Invest. inf.	Inst. 1	Inst. 2	Inst. 3
purch. cost	\$190,500	\$208,000	\$215,000
annual cost	\$8,500	\$6000	\$5000
useful life	7 yrs	7yrs	7yrs

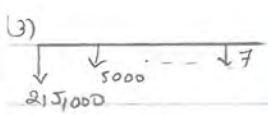
If the MARR = 9% a year. Which inst. is the most economic?



$PWC_1 = 190{,}500 + 8500\,(P/A;9\%,7) = 233280$



$PWC_2 = 208{,}000 + 6000\,(P/A;9\%,7) = 238197$



$PWC_3 = 215{,}000 + 5000\,(P/A,9\%,7) = 240{,}164$

Decision: Investment 1 is the most economic.

Example. A construction firm has taken a project 25 miles away from its head quarters. 15 of the firm's engineers are to be transported daily to manage the project. The project is expected to last 3 years. These are the available options.

1- Buy a minibus
2- Buy 3 small cars
3- Hire a taxi company.

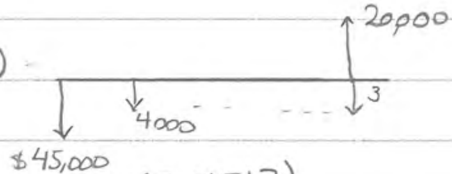
The minibus needs a driver, but the driver wage will be deducted from the engineer's allowances.
The cars will be driven by the engineers but the gasoline's cost will be deducted from the allowances.
The following cost information are available.

1- purchasing cost: 45,000 \$
   annual cost/yr: 4000 \$
   Salvage after 3yrs: 20,000 \$

2- purchasing cost/car: \$20,000
   annual cost/car: \$1000
   salvage value / car: \$10,000

3- The taxi company charges a flat amount 18,000 \$.
The MARR is 10% a year. Which option is the most economic? Use the present worth.

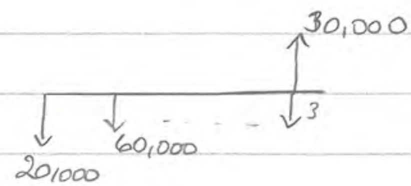
Exclusive service investments
Analysis period: 3 years
(minimize pwc)
Cash flow - For 1^{st} option



$PWC_1 = 45000 + 4000(P/A, 10\%, 3)$
$\quad - 20{,}000(P/F, 10\%, 3)$

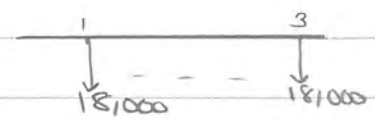
$PWC_1 = 45000 + 4000(2.4869) - 20{,}000(0.7513)$

$PWC_1 = 39{,}921.6\ \$$



$PWC_2 = 20000 + 60{,}000(P/A, 10\%, 3)$
$\quad - 30{,}000(P/F, 10\%, 3)$

$PWC_2 = 44{,}921\$$



$PWC_3 = 18{,}000(P/A, 10\%, 3)$

$PWC_3 = \$44{,}764$

Decision - option 1

About decision on the analysis period

Investments : $I_1, I_2, I_3 \ldots I_k$

Let U_J be the useful life of I_J

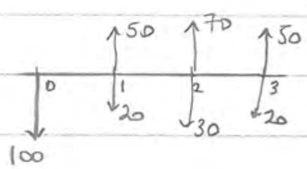
1. If $U_1 = U_2 = U_3 = \ldots = U_k = U$
   Then $T = U$

2. If at least one $U_J \neq$ others then

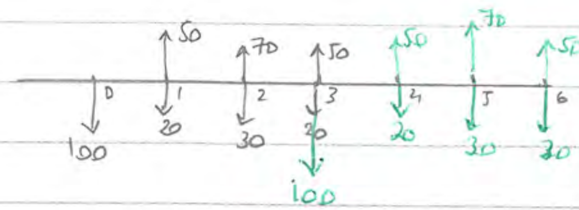
$T = LCM(U_1, U_2, U_3 \ldots, U_k)$ $\qquad$ $T \geq \max\{U_1, U_2 \ldots U_k\}$

If we assume either period, we have to apply the repeatability assumption; cash flows are repeated.

Example.



3 year



6 year (repeated)

Remark

- The analysis period should be reasonable:
  Costs and benefits are highly unlikely to remain fixed over a long time.

- If no analysis period was stated in the question use the LCM period.

By: Ghena Hammour

Consider the two exclusive revenue investments I_1 and I_2.

	I_1	I_2
Initial Cost	\$9500	\$20,000
Annual Benefit	\$4000	\$4,200
Useful life	4yrs	8yrs

If the annual discount rate is 10%, which investment if any is more attractive?
Use present worth.

Revenue invest. : max. positive PW.

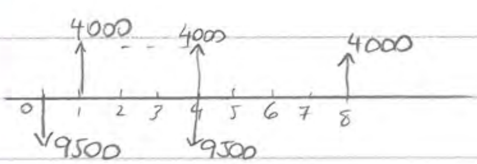
since no analysis period is stated then $T = LCM(8,4)$

$8: \ 4 \times 2 = 2^3$

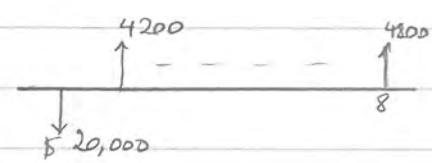
$4: \ 2 \times 2 = 2^2$

$T = LCM(8,4) = 2^3 = 8$ years.

I_1



I_2



PwB, PwC.

$PW_1 = 4000(P/A, 10\%, 8) - 9500(P/F, 10\%, 4) - 9500 = 5351.1$

$PW_2 = 4200(P/A, 10\%, 8) - 20{,}000 = 2406$ \$

Example. A manufacturer is considering one of the two alternatives of buying a machine to produce plastic caps.

	Machine 1	Machine 2
Initial Cost	55,000	65,000
Salvage Value	12,000	15,000
Useful life	4 years	6 years

The manufacturer uses a 7% year discount rate, Which machine is a better buy based on present worth?

service investments: Minimize PWC

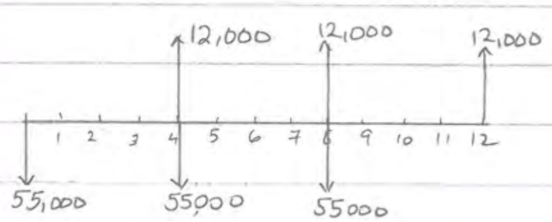
Period not specified so $T = LCM(6,4)$

$6 = 3 \times 2 = 2 \times 3$

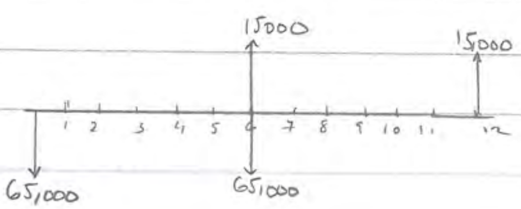
$4 = 2 \times 2 = 2^2$

$LCM(6,4) = 2^2 \times 3 = 12$

machine 1



machine 2



$PWC_1 = 55,000 + 55000(P|F,7\%,4) + 55,000(P|F,7\%,8)$
$\quad -12,000(P|F,7\%,4) - 12,000(P|F,7\%,8) - 12,000(P|F,7\%,12)$

$PWC_1 = 107,500$

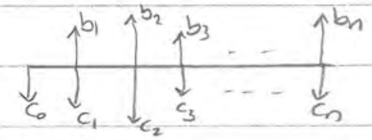
$PWC_2 = 65000 + 65,000(P|F,7\%,6) - 15000(P|F,7\%,6) - 15000(P|F,7\%,12)$

$PWC_2 = \$91,655$

Decision Buy Machine 2. :)

Present worth of projects or investments with infinite life.

(Dams, highways, parks, bridges ...)

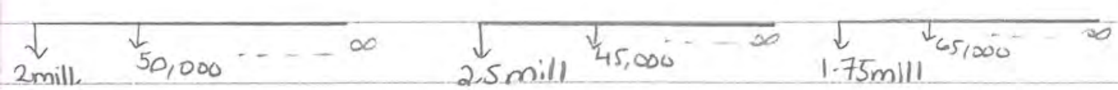


$$PWB = \sum_{J=1}^{\infty} \frac{b_J}{(1+i)^J}, \quad PWC = \sum_{J=0}^{\infty} \frac{C_J}{(1+i)^J}$$

Example. A municipality had 3 plans for the construction of a public park on the same site. All the 3 plans have the same visiting capacity. The estimated costs are:

plan #	Const. Cost	Annual..
1	\$ 2 million	\$50,000
2	\$ 2.5 million	\$ 45,000
3	\$ 1.75 million	\$65,000

All 3 plans are assumed to have infinite life.
The municipality uses a 5% MARR / Year.
Which plan is the most economic. Use present worth.
Service Investment, infinite life.



$PWC_1 = 2\text{mill} + \frac{50,000}{5\%}$ $\quad PWC_2 = 2.5\text{mill} + \frac{45000}{5\%}$ $\quad PWC_3 = 1.75\text{mill} + \frac{65000}{5\%}$

$PWC_1 = 3\text{mill}$ \$ $\quad PWC_2 = 3,400000$ \$ $\quad PWC_3 = \$3,050,000$

Decision Plan 1 1)

Assessment of the Economic worth of Bonds on Present Worth

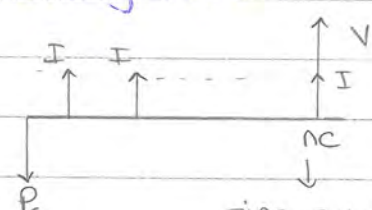
Bonds are issued by governments, banks and companies to raise capital. It is a sort of debt. To the purchasors, a bond is an investment with returns.

A bond in its simplest form has the following features:

- A face value 'V'
- A stated annual interest 'b' called the coupon rate of the bond.
- A duration time on maturatiy of "m" years.
- A purchasing price P_c.

The bond pays interest amounts (dividend) at equal successive "c" subperiod of a year.

The face Value is paid at the maturity time.



since a year contains c subperiods $n = nc$

The dividend $I = \frac{Vb}{c}$

Let i = MARR / subperiod

$PWB = I(P/A, i\%, nc) + V(P/F, i\%, nc)$

$PWC = P_c$

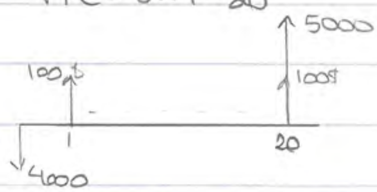
- If $PW \geq 0$ then Bond is worth buying, otherwise its not.

Example.

The central bank had released recently Euro bonds for 5 years.
The stated coupon rate is 8%. The dividend is paid every 3 months.
If your MARR/3 months is 3%. is it worth buying a \$5000 face value bond for \$4000?

$V = \$5000$, $b = 8\%$, MARR/3months $= 3\%$

$P_c = \$4000$ $n = 5yrs$ $c = \frac{12}{3} = 4$ $nc = 5 \times 4 = 20$



$I = \frac{Vb}{c} = \frac{5000 \times 0.08}{4} = 100\$$

$PWB = 100(P/A, 3\%, 20) + 5000(P/F, 3\%, 20)$

$PWC = 4000$

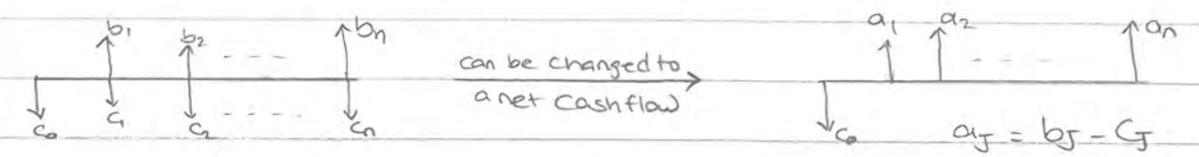
$PW = PWB - PWC = \$250 > 0$

$\therefore$ Bond is worth buying.

The Payback period of a revenue investment.

The payback period "n_p" is roughly the time it takes net benefits to recover the initial cost.

Consider the cash flow of an investment



If $a_n \geq 0$, we can compute a payback period.

There are two types of payback periods.

1. The no return period (0% interest)
2. The discounted period (i = MARR)

Calculation of n_p

① Case of no return (0% interest)

- If the $\sum_{J=1}^{n} a_J < C_0$; net benefits will never recover the initial investment ∴ the investment is bad.

- If $\sum_{J=1}^{k} a_J < C_0 < \sum_{J=1}^{k+1} a_J$ Then;

$$n_p = k + \frac{\left(C_0 - \sum_{J=1}^{k} a_J\right)}{a_{k+1}}$$

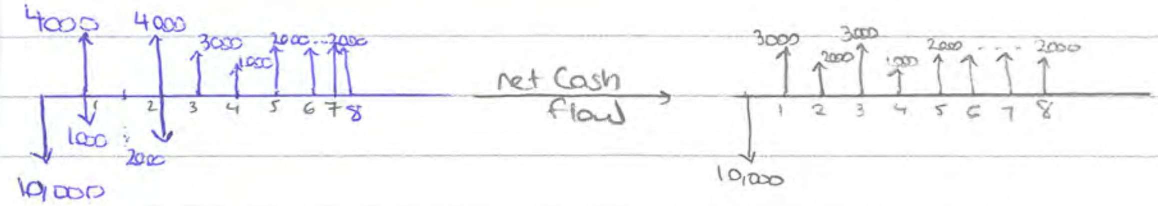
Cummulative sum

this is a period

Initial cost

the "payment" at the upper limit not the cumulative sum!

Example. Calculate the no return payback period for the given investment.



$\sum_{J=1}^{5} a_J = 3000 + 2000 + 3000 + 1000 + 2000 = 11{,}000$

$C_0 = 10{,}000$

C_0 lies between two periods namely 4 where $\sum_{J=1}^{4} a_J = 9000$ and period 5 where $\sum_{J=1}^{5} a_J = 11{,}000$

$$n_p = 4 + \frac{(10{,}000 - 9000)}{2000} = 4.5 \text{ years.}$$

② Case of discounted period.

- If $\sum_{J=1}^{n} PW(a_J) < C_0$; net benefit will never cover initial investment / C_0. ($PW < 0$)

- If $\sum_{J=1}^{k} PW(a_J) < C_0 < \sum_{J=1}^{k+1} PW(a_J)$ Then;

$$n_p = \underbrace{k}_{\text{period}} + \left(\frac{C_0 - \sum_{J=1}^{K} PW(a_J)}{PW(a_{k+1})} \right)$$

(cumulative at that period)

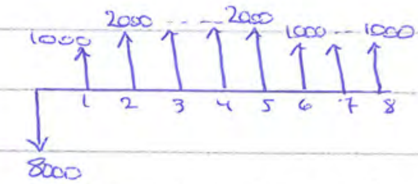
upper limit payment not cumulative

Example

Consider the investment given below. Find the discounted payback period using a discount rate of 5% a year.

J	$PW(a_J)$	$\sum PW(a_J)$
1	952	952
2	1814	2766
3	1728	4494
4	1645	6139
5	1567	7706
6	746	8452

$8000 = C_0$ lies btw these two.



From the table! we formulate it :)

$$n_p = 5 + \frac{8000 - 7706}{746} \simeq 5.4 \text{ years.}$$

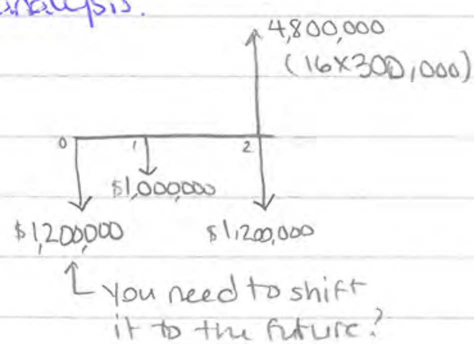
By: Ghena Hammour

Future Worth Analysis.

Future worth Analysis is an extension of present worth;
$FW = FWB - FWC$
In fact, $FW = PW(F/P, i, n)$

Remark.
Future worth is mainly applied to projects that are to be sold upon completion, like:
Apartment buildings, shopping malls, hotels...

Example - An apartment building shall consist of 16 apartments is to be completed in two yrs. All apartments are expected to be sold upon completion at the expected price of 300,000 \$ each. The costs are.
Land cost \$1,200,000
First year cost \$1,000,000
2^{nd} year cost \$1,200,000
At a discount rate of 8% / year, Is the project attractive? Use future worth analysis.



$FWB = 4,800,000$

$FWC = 1,200,000(F/P, 8\%, 2)$
$\quad + 1,000,000(F/P, 8\%, 1)$
$\quad + 1200000 =$

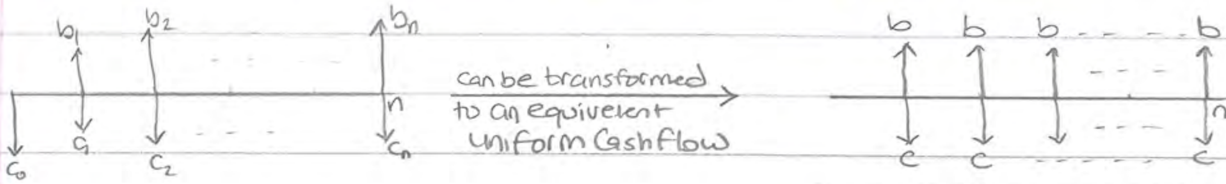
$FW = FWB - FWC$

$FW = 1,180,320$ $\therefore$ project is attractive.

Annual Worth Analysis

- It is similar to present worth but more commonly used.
- In case of many mutually exclusive investments the annual worth has a computational advantage over present worth.

Consider the cash flow of an investment:



c: annual worth of costs
b: annual worth of benefits

How to apply Annual worth method:

Case 1 - Single revenue investment

if $AW \geq 0$ → attractive
if $AW < 0$ → not attractive

Case 2 - Many competing revenue investments

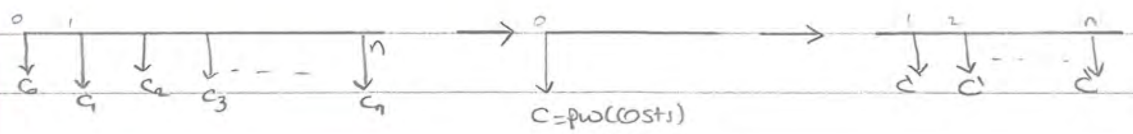
Maximize positive AW. (choose the greatest AW)

Case 3 - Many competing service Investments

Minimize AWC
AWC = AW of all costs − AW (salvage value)

Fact:

The annual worth of an investment computed over a positive integer multiple of its useful life, has the same value as the annual worth value computed over a useful life.



$$C' = C\left[\frac{i(1+i)^n}{(1+i)^n-1}\right]$$ memorize!

Remark

In case of investments (wheather revenue/service) with a different useful life and the adopted analysis period is the LCM, then the annual worth of an investment is computed over its useful life.

Example.

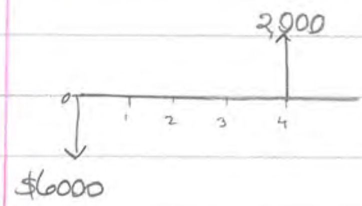
Two pumps are bieng considered for purchasing:

	Pump 1	Pump 2
Initial Cost	\$6000	\$4500
Salvage Value	\$2000	\$900
Useful life	4 yrs	4 yrs

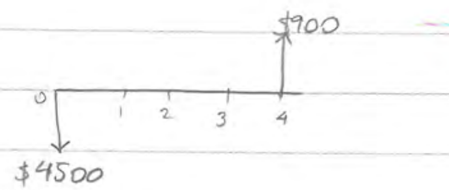
If the MARR is 7% a year, which pump is better to purchase? Use annual worth analysis

Service investment → minimize AWC.

Pump 1



Pump 2



$AWC_1 = 6000(A/P, 7\%, 4) - 2000(A/F, 7\%, 4)$

$AWC_1 = \$1321$ $>$ $AWC_2 = \$1125$

$AWC_2 = 4500(A/P, 7\%, 4) - 900(A/F, 7\%, 4)$

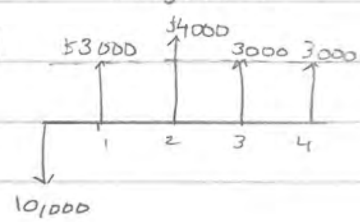
∴ Pump2 is a better purchase.

Example. You are bieng offered to invest 10,000\$ in one of the following products.
The annual benefits are:

Year	Project 1	Project 2
1	\$3000	\$2000
2	\$4000	\$3000
3	\$3000	\$3000
4	\$3000	\$2000
5	—	\$4000

If your MARR = 7% a year. Which project is more attractive. Use annual worth analysis.

Project 1



taking the common 3000 and adding the extra 1000 at n=2

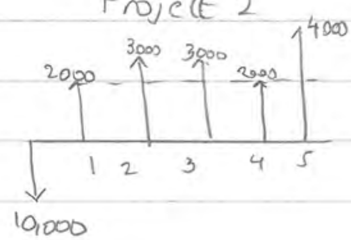
$AWB_1 = 3000 + 1000(P|F, 7\%, 2)(A|P, 7\%, 4)$

$AWB_1 = 3258$

$AWC_1 = 10{,}000(A|P, 7\%, 4) = 2952$

$AW_1 = AWB - AWC_1 = 306$

Project 2



Similar to above

$$AWB_2 = 2000 + 1000[(P|F, 7\%, 2) + (P|F, 7\%, 3) + 2(P|F, 7\%, 5)](A|P, 7\%, 5)$$

$AWB_2 = 2760$

$AWC_2 = 10{,}000(A|P, 7\%, 5) = 2432$

$AW_2 = 328$

∴ Project 2 is more attractive :)

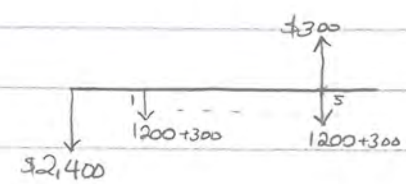
Example

A pump is required for 10yrs at a remote location. The pump can be driven by an electric motor, if a power line is extended to the site, otherwise, a gasoline engine will be used. Use annual worth and a 10% MARR to determine based on the following information.

	Gasoline	Electric
First cost	\$2,400	\$6000
Annual operating cost	\$1,200	\$750
Annual maintanence cost	\$300	\$50
Salvage value	\$300	\$600
Useful life	5yrs	10yrs

- Service Investment ; minimize AWC
- Analysis period $T = LCM(5,10) = 10$ yrs but we need here to compute over a useful life since this is AW analysis.
- MARR = 10%

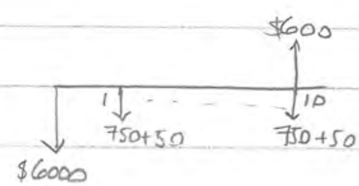
Gasoline



* Gasoline

$$AWC_1 = 2{,}400\,(A/P, 10\%, 5) + 1500$$
$$-300\,(A/F, 10\%, 5)$$
$$AWC_1 = \$2084$$

Electric



* Electric

$$AWC_2 = 6000\,(A/P, 10\%, 10) + 800$$
$$-600\,(A/F, 10\%, 10)$$
$$AWC_2 = \$1739$$

Decision → Electric!

By: Ghena Hammour

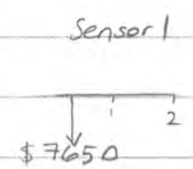
A mechanical Engineer is considering three types of pressure sensors. With the following costs:

	Sensor 1	Sensor 2	Sensor 3
First Cost	\$7650	\$12,900	\$16,000
Annual main. Cost	0	\$1000	\$700
Salvage Value	0	\$2,500	\$1000
Useful Life	2yrs	4yrs	6yrs.

At a discount rate of 12% a year. Which sensor is the most economic to buy? Use Annual Worth.

Service investment: minimize AWC

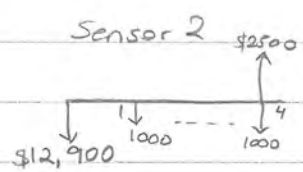
* Sensor 1



$AWC_1 = 7650\,(A/P, 12\%, 2)$

$AWC_1 = \$4527$

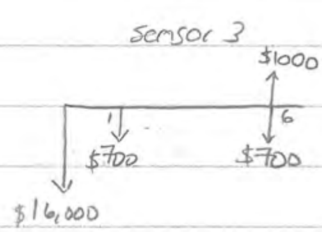
* Sensor 2



$AWC_2 = 12900\,(A/P, 12\%, 4) + 1000$
$\quad - 2500\,(A/F, 12\%, 4)$

$AWC_2 = \$3724$

* Sensor 3



$AWC_3 = 16{,}000\,(A/P, 12\%, 6) + 700$
$\quad - 1000\,(A/F, 12\%, 6)$

$AWC_3 = \$3769$

Decision: Sensor 2.

Some equipment will be installed in a warehouse that a firm for 7yrs. There are two exclusive alternatives.

	A	B
Initial Cost	\$100	\$150
Annual Benefits	\$55	\$61
Useful life	3yrs	4yrs

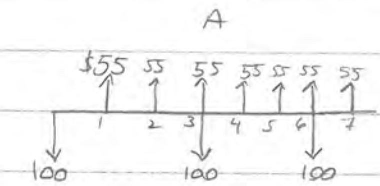
The equipment has no salvage value anytime after installation.

If the MARR is 10% a year. which alternative if any is attractive? Use annual worth analysis and an analysis period of 7 yrs.

Revenue Investment → maximize positive AW.

$T = 7$ yrs.

* A



$AWB_1 = 55\ \$$

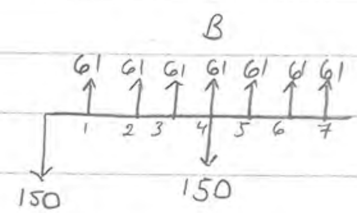
X $AWC_1 = 100(A/P, 10\%, 7) + 100(A/F, 10\%, 5)$
$+100(A/F, 10\%, 5)$

or $AWC_1 = [100 + 100(P/F, 10\%, 3) + 100(P/F, 10\%, 6)](A/P, 10\%, 7)$

$AWC_1 = \$47.52$

$AW_1 = 55 - 47.52 = \$7.48$

* B



$AWB_2 = 61\ \$$

$AWC_2 = [150 + 150(P/F, 10\%, 4)](A/P, 10\%, 7)$

$AWC_2 = \$51.85$

$AW_2 = 61 - 51.85 = \$9.15$

Decision → B

Note

if we had a salvage value and we need to extend/repeat the cash flow, we will need inf. about it.

Lecture 16.

By: Ghena Hammour

Rate of Return Analysis Method (ROR, IRR)

Definition:

The rate of return of a single revenue investment is the unique positive interest rate which balances the cash flow.

Mathematically, it is the unique positive root of the eqn:

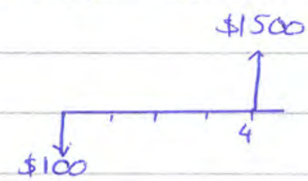
$PW(i) = 0$ or $FW(i) = 0$ or $AW(i) = 0$

$-100\% < i < \infty$ is a real variable.

How to solve for the rate of return value?

1- Directly.

Example. Find the rate of return for the following investment.



$PW(i) = \frac{1500}{(1+i)^4} - 100$

$PW(i) = 0$

$(1+i)^4 = 1.5$

$\Rightarrow ROR = 10.67\%$

2- Using a numerical method

3- By trial and error using interpolation.

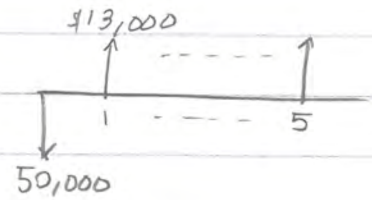
Note!

ROR can only be applied on revenue investments :)

Example.

An investment of \$50,000 will return annual benefits of \$13,000 for the next 5 yrs.

Find its rate of return.



$PW(i) = 13{,}000(P/A, i, 5) - 50{,}000$

$PW(i) = 0$

$13{,}000(P/A, i, 5) = 50{,}000$

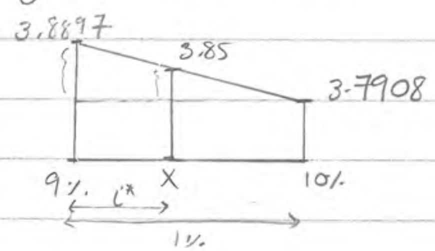
$(P/A, i, 5) = 3.85$

Check the interest table to corner/bound the value 3.85..

$(P/A, 9\%, 5) = 3.8897$

$(P/A, 10\%, 5) = 3.7908$

Using interpolation



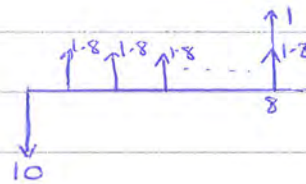
$$\frac{0.01 - X}{0.01} = \frac{3.85 - 3.7908}{3.8897 - 3.7908}$$

$X = 0.004$

$ROR = (0.09 + 0.004) \times 100 = 9.4\%$

lecture 17

Example. Consider the investment in ($ million)
Find its rate of return.



$pw(i) = 1.8\,(\underline{P/A, i, 8}) + (1)(\underline{P/F, i, 8})$

$pw(i) = 0$

Here we have two unknown factors,
So we try interest rates from the interest tables.
BUT which interest value do we start with?
Your first trial is found from this:

$$\frac{\left(\dfrac{\text{Sum of upper flow} - \text{Sum of lower flow}}{\text{Sum of lower flow}}\right) \times 100}{n} + 2\%$$

First trial $\quad i = \left(\dfrac{\dfrac{(1.8 \times 8 + 1) - 10}{10}}{8}\right) \times 100\% + 2\% = 8.75\%$

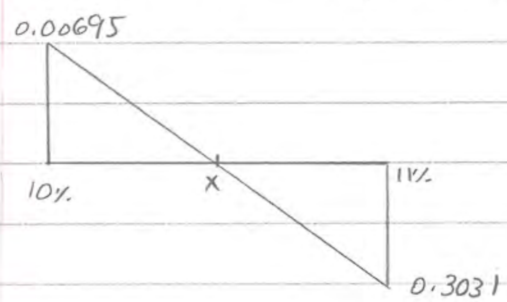
$pw(9\%) = 0.4649$

$pw(10\%) = 0.00695$

$pw(11\%) = -0.3031$ stop when negative!

$\Rightarrow$ ROR is between 10% and 11%.

Now we use interpolation:



$$\frac{X}{0.01 - X} = \frac{0.00695}{0.3031} \Rightarrow X = 0.00186$$

$ROR = (0.1 + 0.00186) \times 100 \simeq 10.186\ \%$

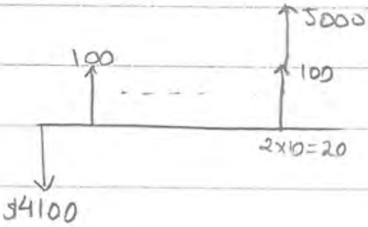
Example. Calculating the rate of return of a Bond.

A Bond with a face value of \$5000 was purchased by an investor for 4,100. The bond is due in 10yrs. The coupon rate is 4% and dividend is paid every 6 months. If the investor kept the bond till maturity, what rate of return the investor made per 6months and per year?

$V = 5,000 \qquad b = 4\% \qquad c = 2 \qquad P_c = 4,100\,\$$

Dividend : $I = \dfrac{(0.04)(5000)}{2} = 100$

$PW(i) = 100\,(P/A,\, i,\, 20) + 5000\,(P/F,\, i,\, 20) - 4100$

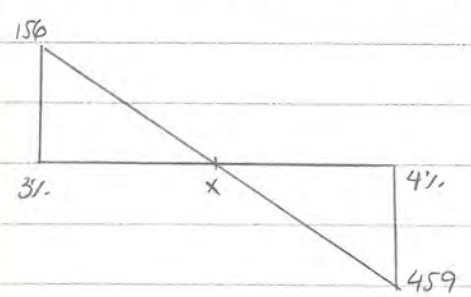


We have two unknown factors;
So we solve by trial and error

Ask first try $i = \left(\dfrac{\frac{(100\times 20 + 5000) - 4100}{4100}}{2e} \right) \times 100 + 2\% = 5.53\%$

$PW(3\%) = 156$
$PW(4\%) = -459$

RoR is btw 3% and 4%. Now we use interpolation:



$\dfrac{x}{0.01 - x} = \dfrac{156}{459} \Rightarrow x = 0.00253$

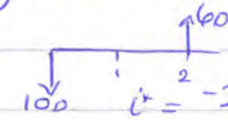
$ROR \simeq (0.03 + 0.00253) \times 100 = 3.253\%$ per 6months.

Annual $ROR = 3.253\% \times 2 = 6.506\%$

By: Ghena Hammour

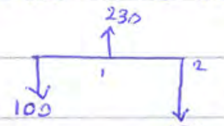
Some features about the ROR.

- Sometimes $pw(i) = 0$ has a single negative root which means the investment is bad.



$i^* = -22.6\%$

- Sometimes $pw(i)$ has multiple positive roots.

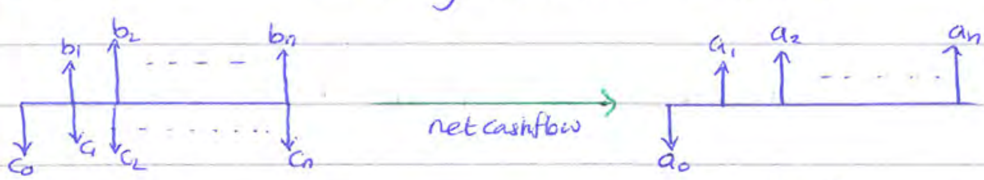


$pw(i) = 0 \begin{cases} i_1 = 10\% \\ i_2 = 20\% \end{cases}$

In this case we have two options, where
- We either use another method
- or we use a modified ROR method (to be discussed)

- Testing for the existence of the ROR:

Consider the arbitrary revenue investment:



$a_j = b_j - c_j \quad ; j \geq 1$

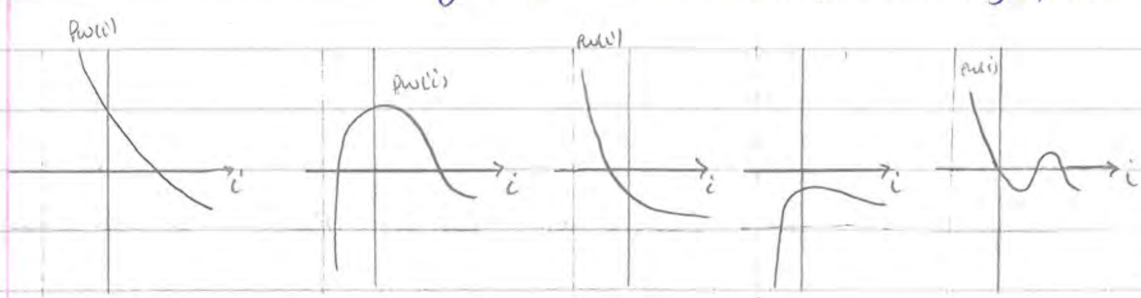
$a_0 = c_0$

$\lim_{i \to \infty} pw(i) = a_0$

Now, $pw(i) = a_0 + \dfrac{a_1}{1+i} + \dfrac{a_2}{(1+i)^2} + \dots + \dfrac{a_n}{(1+i)^n} \; ; -1 < i < \infty$

$\lim_{i \to -100\%} pw(i) \begin{cases} \to \infty \text{ if } a_n > 0 \\ \to -\infty \text{ if } a_n < 0 \end{cases}$

The function has: $\begin{cases} i = -100\% \text{ as a vertical asymptote} \\ y = a_0 \text{ as a horizontal asymptote.} \end{cases}$



unique ROR values — unique ROR values — No ROR values — No ROR values — multiple +ve roots

* The equivalent polynomial equation to $pw(i) = 0$
  let $1 + i = x$
  Then $pw(i) = 0$ is equivalent to the polynomial equation:
  $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_n = 0$

  $\Rightarrow$ The number of roots of $pw(i) = 0$ equals to the number of positive roots of the polynomial equation.

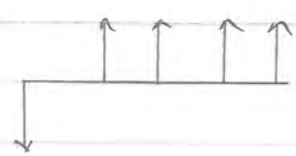
* Now by Decarte's rule of signs, the number of positive roots of a real polynomial is less or equal to the number of sign changes among the coefficients.

  Ex: consider the equation:
  $$x^5 + 2x^4 + x^3 - x^2 + 2x + 1 = 0$$
  2 sign changes
  $\therefore$ the number of positive roots ≤ 2.

  Definition: Consider the net cash flow of an investment (I). The Total Flow of (I) is:
  $TF(I) = \sum_{j=0}^{n} a_j$
  observe that the total flow of the investment is
  $TF(I) = pw(0)$

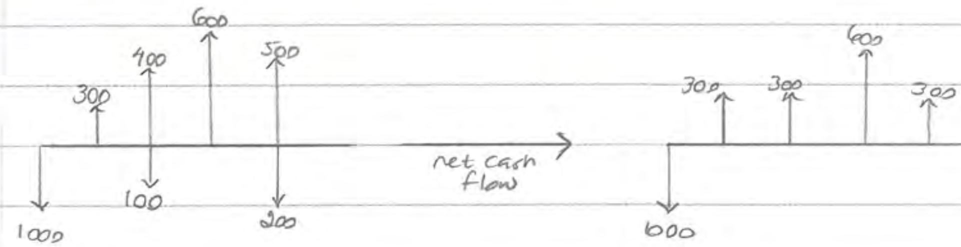
* An investment is a simple investment if its net cash flow has exactly one sign change.

  Example: This is a simple investment $\Rightarrow$



Theorem/Fact:
A simple investment has a unique ROR value iff its total flow $TF(I) > 0$.

Example: Consider the investment.

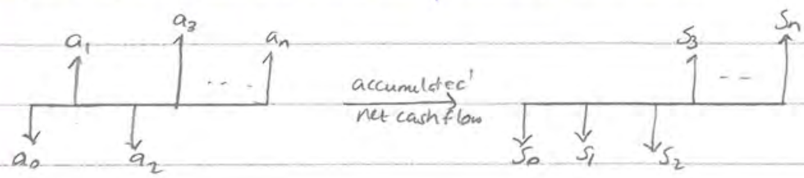


$TF > 0 \Rightarrow$ the investment has a unique ROR value.

If the investment is not simple then it is called non-simple.

Remark:
Non simple investments are a potential source for the existence of multiple positive roots of $pw(i)=0$.

* The accumulated net cash flow test or NORSTROMS test for non simple investment:
Consider the non simple investment:



$S_0 = a_0$
$S_1 = a_0 + a_1$
$S_2 = S_1 + a_2$
$\vdots$
$S_n = S_{n-1} + a_n$

Now, if the accumulated net cash flow has exactly one sign change, then the investment has a unique ROR value.

Example.

Consider the net cash flow for the following non simple investment. Use NORSTROM's test to investigate the existence of a unique ROR value.

Period	A	B	C
0	-100	-100	-100
1	50	150	50
2	100	-100	0
3	-25	-150	200
4	-10	-50	-50

From Accumulated Net cash flow:

Period	A	B	C
0	-100	-100	-100
1	-50	50	-50
2	50	-50	-50
3	25	-200	-150
4	15	-250	100

A: has a unique ROR value (1 sign change)

B: inconclusive (2 sign changes)

C: has a unique ROR value (1 sign change)

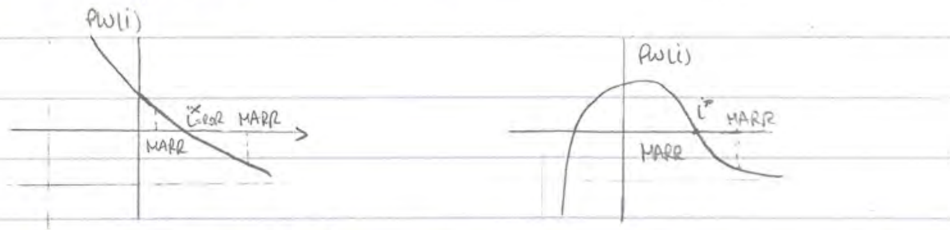
Lecture 19

By: Ghena Hammour

How to apply the Rate of Return Method?

Case of a single investment:

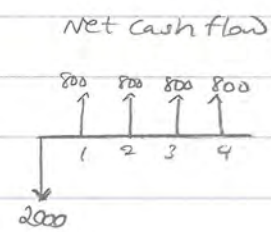
If the investment has a unique ROR value then compute it and:



$\begin{cases} \text{if MARR} \leq \text{ROR} & : \text{investment is attractive} \\ \text{if MARR} > \text{ROR} & : \text{investment is not attractive.} \end{cases}$

Example:

An investment of \$2000 will return annual benefits of \$800 for the next 4 yrs.
If the MARR is 17% a year. Is the investment attractive? Use the ROR method.



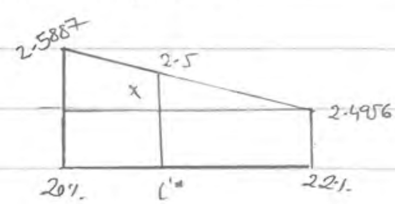
× investment is simple since TF>0
⇒ Unique ROR value.

$PW(i) = 800(P/A, i, 4) - 2000$

$PW(i) = 0$

$(P/A, i, 4) = 2.5$ check interest table, use interpolation

$20\% < i^* < 22\%$.



$$\frac{2.5 - 2.4936}{2.5887 - 2.4936} = \frac{0.02 - x}{0.02}$$

$x = 0.0186$

$ROR = (0.2 + 0.0186) \times 100 = 21.86\% > MARR$

∴ The investment is attractive.

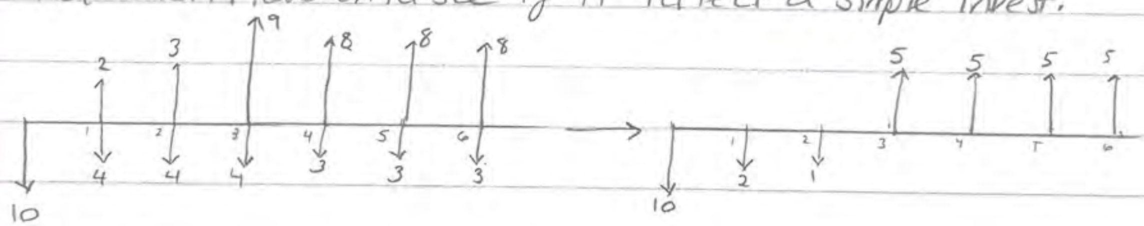
Example – The expected costs and benefits (in mill. $) of a $10 million investment are:

Year	Costs	Benefits
1	4	2
2	4	3
3	4	9
4	3	8
5	3	8
6	3	8

a) Find the ROR value of this investment.

b) Is the investment attractive if the MARR/year is 8%, 10%, 10.5%, 12%?

a) We start by drawing the cash flow and then the net cash flow and see if TF reflect a simple invest.



We observe a TF with only one sign change, hence this is a simple investment with a unique ROR value.

$PW(i) = -10 - 2(P/F,i,1) - 1(P/F,i,2) + 5(P/A,i,4)(P/F,i,2)$

- Now, we need to determine the "first" guess of i.

$$i' = \left(\frac{(5+5+5+5)-(10+2+1)}{(10+2+1)}\right)\cdot\frac{1}{6}\times 100 + 2\% = 0.0896\times 100 + 2\% \simeq 10.97\%$$

Start by $i = 10\%$, from interest tables:

$PW(10\%) = -10 - 2(P/F,10\%,1) - 1(P/F,10\%,2) + 5(P/A,10\%,4)(P,F,10\%,2)$

$PW(10\%) = -10 - 2(0.9091) - (0.8264) + 5(3.1699)(0.8264)$

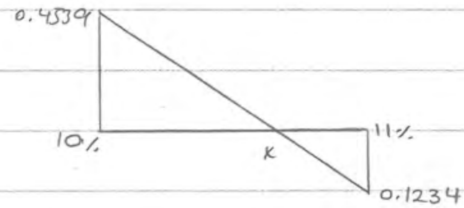
- $PW(10\%) = 0.4534268$

$PW(11\%) = -10 - 2(P/F,11\%,1) - 1(P/F,11\%,2) + 5(P/A,11\%,4)(P/F,11\%,2)$

$PW(11\%) = -10 - 2(0.9009) - (0.8116) + 5(3.1024)(0.8116)$

- $PW(11\%) = -0.02386 \simeq -0.1234$

Now we have 2 values and hence we can use interpolation to find ROR.



$PW(10\%) = 0.4539$

$PW(11\%) = -0.1234$

$$\frac{x}{0.01 - x} = \frac{0.4539}{0.1234}$$

$x = 0.00786$

$\Rightarrow i^* = ROR = (0.1 + 0.00786) \times 100 = 10.786\%$

b) For MARR = 8% ROR > MARR ⇒ Attractive

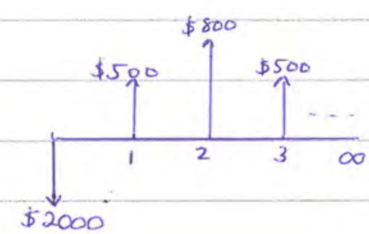
MARR = 10% " "

MARR = 10.5% " "

For MARR = 12% ROR < MARR ⇒ Not attractive.

Take home.

Find the ROR of this investment



The Rate of Return method: Multiple alternatives

Incremental or differential Investments:

Consider two exclusive revenue investments I_1 and I_2 with the same analysis period.

Assume that the initial cost of I_2 is higher than the initial cost of I_1.

I_2 is considered as an increment of I_1.

Now, the investment $(I_2 - I_1)$ of the cash flow is called the incremental investment.

Let us assume that at the MARR,

$pw(I_1) > 0$ and $pw(I_2) > 0$.

$pw(I_2 - I_1) = pw(I_2) - pw(I_1)$

→ if $pw(I_2 - I_1) > 0$ then I_2 is prefered to I_1

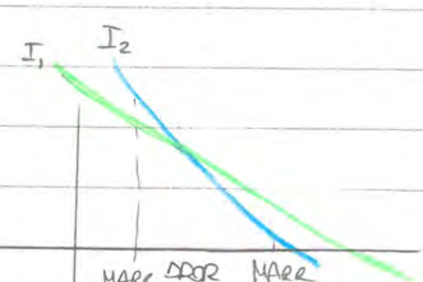
Otherwise, I_1 is prefered to I_2.

Now, if I_1 and I_2 have unique rates of Return and $I_2 - I_1$ has a unique rate of return call it ΔROR

then, if $\Delta ROR > MARR$

$\Rightarrow I_2$ is prefered to I_1

otherwise, I_1 is prefered to I_2.



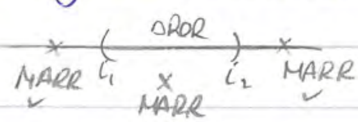
Incremental ROR method:

Case of two exclusive revenue investments.

Consider two mutually exclusive revenue investments I_1 and I_2 where:

- They have the same analysis period.
- Their rates of Return $\geq$ MARR
- The initial cost of I_2 > initial cost of I_1.

We form the incremental investment $I_2 - I_1$
if $I_2 - I_1$ has a unique ROR value call it ΔROR then locate it (bracket it) between two interest values: $i_1 < \Delta ROR < i_2$ such that the MARR value doesn't belong to the interval i_1, i_2 $(MARR \notin (i_1, i_2))$

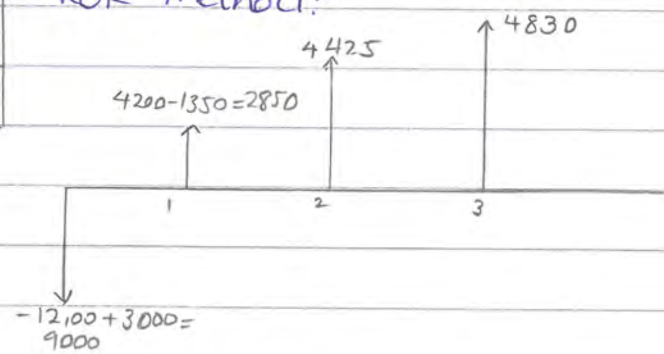


Then,
if $\Delta ROR > MARR$; I_2 is better than I_1.
if $\Delta ROR < MARR$; I_1 is better than I_2.

Example. Consider the investments.

Period	A	B
0	-3000	-12000
1	1350	4200
2	1800	6225
3	1500	6330
ROR	25.1%	17.43%

If the MARR is 10%. Which investment is more attractive based on the ROR method?



Form the incremental investment

$\Rightarrow B - A$

Since TF > 0, ⇒ unique ΔROR

$PW(i) = -9000 + 2850(P/F, i, 1) + 4425(P/F, i, 2) + 4830(P/F, i, 3)$

$PW(i') = 0$

first trial of i

$i = \left(\frac{(2850 + 4425 + 4830) - 9000}{9000}\right) \cdot \frac{1}{3} \times 100 + 2\% = 11.5\% + 2\% = 13.5\%$

$PW(14\%) = -9000 + 2850(0.8772) + 4425(0.7695) + 4830(0.6750)$

$PW(14\%) = 165.3075$

$PW(15\%) = -9000 + 2850(0.8696) + 4425(0.7561) + 4830(0.6575)$

$PW(15\%) = -0.1725$

$14\% < \Delta ROR < 15\%$

$\Rightarrow$ B is more attractive. since MARR $\notin$ (14%, 15%), MARR = 10%

Example. Consider the investment.

	A	B
Initial Cost	10,000	13,000
Annual Benefits	2,500	3,500
Useful life	5yrs	5yrs.

If the MARR is 12% a year. Which investment if any is more attractive based on the ROR method?

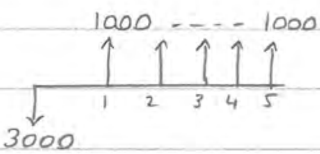
- We are solving this bypassing the calculation of ROR.

From the incremental investment: B-A.

$PW(i) = -3000 + 1000(P/A, i, 5)$

$PW(i) = 0$

$(P/A, i, 5) = \frac{3000}{1000} = 3$



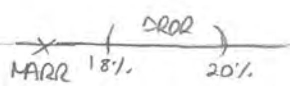
Checking interest tables.

simple, TF>0 $\Rightarrow$ unique ΔROR

$(P/A, 18\%, 5) = 3.1272$

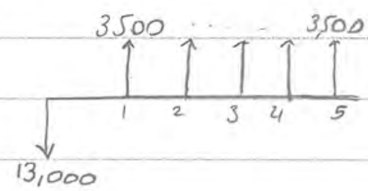
$(P/A, 20\%, 5) = 2.9906$

$\Rightarrow 18\% < \Delta ROR < 20\%$



$\therefore$ Select B.

We still need to calculate ROR of B.



$PW(i) = -13{,}000 + 3500(P/A, i, 5)$

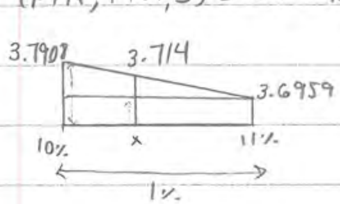
$(P/A, i, 5) = \frac{13000}{3500} = 3.714285714$

looking at interest tables:

Simple, TF > 0 $\Rightarrow$ Unique ROR value.

$(P/A, 10\%, 5) = 3.7908$

$(P/A, 11\%, 5) = 3.6959$



$\frac{3.714 - 3.6959}{3.7908 - 3.6959} = \frac{1\% - X}{1\%} \Rightarrow X = 8.0927 \times 10^{-3}$

$ROR_{(B)} = 10\% + 0.8092\% = 10.8\% < MARR$

$\Rightarrow$ Both investments are <u>not</u> attractive.

Applying the ROR method to many investments.

Given "k" investments : $I_1, I_2, \ldots, I_k$ with the same useful life.

Option 1

1. Calculate ROR of each investment
2. Remove the ones with ROR < MARR
3. Suppose there remains j investments, then arrange these investments starting with the smallest initial cost (in an increasing fashion): $I_1 \ldots\ldots I_j$

   Then Apply incremental ROR $\underbrace{I_1\ I_2}$ — select one of them as usual, call it S_1, then do incremental on $\underbrace{S_1\ I_3}$ — select one ---- and so on until you reach S_{j-1}.

Option 2

1. Arrange investments according to there initial cost in an increasing fashion (starting with the smallest) and
2. Apply incremental ROR same way as in option 1 (3)
   (remember, here you haven't eliminated any yet since you didn't calculate the ROR of each)
3. Once you reach S_{k-1} then you calculate its ROR
   - if ROR of S_{k-1} > MARR → select it
   - if ROR of S_{k-1} < MARR → None of the investments is attractive.

Remark.

- Option 2 can only be used in case of same useful life otherwise use option 1 with LCM.

Example. Consider the 4 mutually exclusive revenue investments.

	A	B	C	D
Initial Cost	\$75	\$50	\$55	\$86
Annual Benefits	\$16	\$12	\$13	\$18
Useful Life	10yrs	"	"	"

If the MARR is 8% a year, which investment if any is most attractive. Based on ROR method.

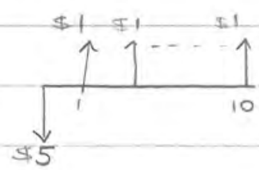
Same useful life, we can use option 2

(1) Arrange investments according to an increasing initial cost:

	B	C	A	D
initial Cost	50	55	75	86
Annual Benefits	12	13	16	18

(2) Apply incremental Analysis as described on pairs.

* starting by Incremental ROR C−B



simple, TF > 0 ⇒ Unique ΔROR

$AW(i) = -5(A/P, i, 10) + 1$

$5(A/P, i, 10) = +1$

$(A/P, i, 10) = \frac{1}{5} = 0.2 \Rightarrow \frac{i^*(1+i^*)^{10}}{(1+i^*)^{10}-1} = 0.2$

$i^* = 15.098 = \Delta ROR$

Another way is to corner the value using interest tables

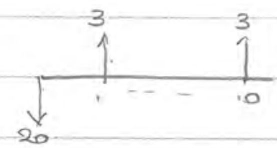
$(A/P, i, 10) = 0.2 \rightarrow$ at 15% $\rightarrow 0.19925$

at 16% $\rightarrow 0.20690$

so $15\% < \Delta ROR < 16\%$ and our MARR is 8%.

Hence, we select C (since ΔROR > MARR) and automatically B is removed.

* Now apply incremental ROR on A–C



Simple, TF > 0 ⇒ unique ΔROR

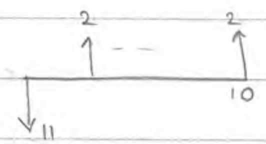
$AW(i) = -20(A/P, i, 10) + 3$

$AW(i) = 0$

$(A/P, i, 10) = 0.15 \Rightarrow \dfrac{i^*(1+i^*)^{10}}{(1+i^*)^{10}-1} = 0.15$

ΔROR = 8.144% > MARR ⇒ select A, remove B,

* Now apply incremental ROR on D–A



Simple, TF > 0 ⇒ unique ΔROR

$AW(i) = -11(A/P, i, 10) + 2$

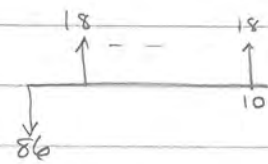
$AW(i) = 0$

$(A/P, i, 10) = \dfrac{2}{11} \Rightarrow \dfrac{i(1+i)^{10}}{(1+i)^{10}-1} = \dfrac{2}{11}$

ΔROR = 12.66% > MARR → select D

(3) Now we calculate ROR of our final selection which is D.

Cash flow of D



Simple, TF > 0 ⇒ Unique ROR value

$AW(i) = -86(A/P, i, 10) + 18$

$AW(i) = 0$

$(A/P, i, 10) = \dfrac{18}{86} \Rightarrow \dfrac{i(1+i)^{10}}{(1+i)^{10}-1} = \dfrac{18}{86}$

ROR = 16.31% > MARR ∴ D is the most attractive.

Example. Consider the investments.

Period	A	B
0	-\$100	-\$150
1	\$30	\$43
2	\$30	\$43
3	\$30	\$43
4	\$30	\$43
5	\$30	\$43

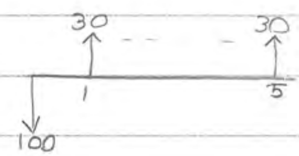
Based on the ROR method, which investment should be selected if the MARR is

a) 6%. b) 8%. c) 10%.

Here we will use option 1

(1) Calculate The ROR value of each investment.

* Calculating ROR of A:



simple, TF > 0 ⇒ unique ROR

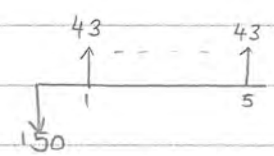
$AW(i) = -100(A/P, i, 5) + 30$

$AW(i) = 0$

$$(A/P, i, 5) = \frac{30}{100} \Rightarrow \frac{i^*(1+i^*)^5}{(1+i^*)^5 - 1} = \frac{30}{100}$$

$ROR_A = 15.238\%$

* Calculating ROR of B:



simple, TF > 0 ⇒ unique ROR

$AW(i) = -150(A/P, i, 5) + 43$

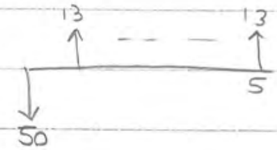
$AW(i) = 0$

$$(A/P, i, 5) = \frac{43}{150} \Rightarrow \frac{i^*(1+i^*)^5}{(1+i^*)^5 - 1} = \frac{43}{150}$$

$ROR_B = 13.33\%$

(2) Both ROR_A, ROR_B are greater than MARR so we don't remove any and proceed to next step.

(3) Apply incremental ROR on B − A



simple, TF > 0 ⇒ unique ROR

$AW(i) = -50(A/P, i, 5) + 13$

$AW(i) = 0$

$(A/P, i, 5) = \frac{13}{50} \Rightarrow \frac{i^*(1+i^*)^5}{(1+i^*)^5 - 1} = \frac{13}{50}$

$i^* = \Delta ROR = 9.43\%$

Now in case:

a) $MARR = 6\% \Rightarrow \Delta ROR > MARR \rightarrow$ select B

b) $MARR = 8\% \Rightarrow \Delta ROR > MARR \rightarrow$ select B

c) $MARR = 10\% \Rightarrow \Delta ROR < MARR \rightarrow$ select A.

Example Consider 4 mutually exclusive invest.

	A	B	C	D
Initial Cost	\$1000	\$800	\$600	\$500
Annual Benefit	\$130	\$127	\$150	\$122
Useful Life	8yrs	8yrs	8yrs	8yrs.

If the MARR is 8% a year, which investment if any is the most attractive.

Using option 1

(1) Calculate ROR of each

* $ROR_A =$

$PW(i) = -1000 + 130(P/A, i\%, 8) = 0$

$PW(i) = 0$

$ROR_A \simeq 0.75\% < MARR \therefore$ remove

* $ROR_B =$

$PW(i) = -800 + 127(P/A, i, 8) = 0$

$ROR_B \simeq 4.8\% < MARR \therefore$ remove

* $ROR_C =$

$PW(i) = -600 + 150(P/A, i, 8) = 0$

$ROR_C \simeq 17.9\% > MARR \therefore$ keep.

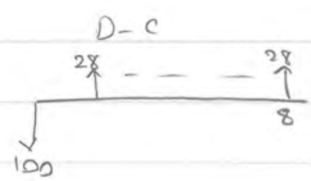
* ROR_D

$PW(i) = -500 + 122(P/A, 8\%, 8) = 0$

$ROR_D \simeq 17\% > MARR \therefore$ keep.

(2) Arrange investments D C and calculate ΔROR

* ΔROR



$PW(i) = -100 + 28(P/A, i, 8)$

$100 = 28(P/A, i, 8)$

$\Delta ROR = 22.46\% > MARR$

$\therefore$ Select D

Multiple Alternatives with different Useful lives

- Compute ROR of each and remove the investments with ROR < MARR.
- Arrange the remaining investments according to increasing initial cost.
- Apply systematic elemination using incremental method., Use AW method to get the advantage of computing only over useful life. $(AW_{J+1} - AW_J)$

Example: Consider the 4 mutually Exclusive investments.

	A	B	C	D
Initial Cost	\$1742	\$1440	\$1160	\$1030
Annual Benefits	\$400	\$600	\$500	\$450
Useful life	6yrs	3yrs	3yrs	3yrs

If the MARR is 13% per year, which investment if any is more attractive. Based on ROR method

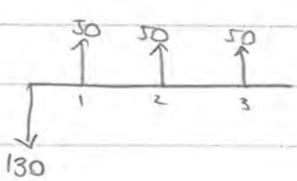
(1) different useful life, calculate ROR value of each, remove ones ROR < MARR

* $ROR_A = 10\% < MARR$ ∴ remove * $ROR_B = 12\% < MARR$ ∴ remove

* $ROR_C = 14\% > MARR$ ∴ keep * $ROR_D = 15\% > MARR$ ∴ keep

(2) Arrange remaining investments, calculate ΔROR using AW method. However, here the remaining investments have same useful life, so we can use any method we want

(3) C − D net cash flow:



$PW(i) = -130 + 50(P/A, i, 3) = 0$

$\frac{130}{50} = (P/A, i, 3) = 2.6$

$\Delta ROR = 7.5\% < MARR$

∴ Select D

The Benefit- Cost ratio method. (B/c)

$\frac{B}{C} = \frac{PWB}{PWC} = \frac{FWB}{FWC} = \frac{AWB}{AWC}$ (in the broad revenue)

Features of the Method:

- It is applicable to revenue investments for the private sector.
- It is mainly applicable to public sector projects.

Some major differences between private and public sector investments.

	Private	Public
Investment size	small	large
Duration Time	short	long
funding	Banks, financial corp.	Tax pay, money and Bonds
MARR	High	Low

There are two ratios:

1. The conventional ratio:

$$\frac{B}{C} = \frac{B - D}{C + (M \text{ and } O) - S}$$

where; B: PW / AW of Benefits.
D: PW / AW of disbenefits.
C: PW / AW of initial cost
MandO: PW / AW of operation and maintainence.
S: PW / AW of salvage value.

Remark:
All parameters in the formula are either in PW or AW.

2. The modified Ratio

$$\frac{B}{C} = \frac{B - D - (M \text{ and } O)}{C - S}$$

Remark:

Both ratios are either <1 or >1.

Example.

A public project has the following information:

AW (initial Cost) = 1,200,000 \$

AW (benefits) = 1,300,000 \$

AW (disbenefits) = 100,000 \$

AW (M and O) = 300,000 \$

AW (Salvage Value) = \$0.

Compute both ratios.

1. Conventional ratio:

$$\frac{B}{C} = \frac{B - D}{C + (M \text{ and } O) - S} = \frac{1,300,000 - 100,000}{1,2000 + 300,000 - 0} = 0.8 < 1$$

2. The Modified ratio:

$$\frac{B}{C} = \frac{B - D - (M \text{ and } O)}{C - S} = \frac{1,300,000 - 100,000 - 300,000}{1,200,000 - 0} = 0.75$$

How to apply the Benefit Cost ratio Method?

- Case of single investments.

At the MARR compute $\frac{B}{C}$ ratio,

$$\begin{cases} \text{if } \frac{B}{C} \geq 1 \longrightarrow \text{attractive} \\ \text{if } \frac{B}{C} < 1 \longrightarrow \text{Not attractive} \end{cases}$$

Example:

Consider the information about a public project.

PW (Benefits) = \$3,700,000

AW d(Benefits) = \$60,000

First Cost = \$2,200,000

(M and O) / year = \$250,000

Life time = 15yrs

If the MARR is 6% a year, is the project attractive based on the Benefit-Cost ratio.

! You only need to calc. one ratio.

All values shall be either PW or AW.

AW (Benefits) = 3,700,000 (A/P, 6%, 15) = \$380,952

AW (first Cost) = 2,200,000 (A/P, 6%, 15) = \$226,512.

1) Conventional ratio:

$$\frac{B}{C} = \frac{380,952 - 60,000}{226,512 + 250,000} = 0.673 < 1$$

2) Modified ratio:

$$\frac{B}{C} = \frac{380,952 - 60,000 - 250,000}{226,512} = 0.313 < 1$$

$\Rightarrow$ Project is not attractive since $\frac{B}{C} < 1$.

- Case of many investments (Mutually Exclusive)

A motivating example:
Consider the two investments.

	I_1	I_2
PWB	\$ 400	\$900
PWC	\$ 200	\$ 500
B/C ratio	2	1.8

Use incremental method, such that I_2 is referred to I_1.
If we consider the incremental investment $I_2 - I_1$ and calculate its Benefit-Cost-Ratio:

$$\frac{\Delta B}{\Delta C} = \frac{900-400}{500-200} = 1.667 > 1$$

So I_2 is prefered to I_1.

Incremental Benefit - Cost ratio:

- Case of two mutually exclusive investments

Consider the exclusive investments I_1, I_2
Where both have the same analysis period, and
both have Benefit - Cost ratio > 1

Assumme $PWC_2 > PWC_1$
or $AWC_2 > AWC_1$

Then, we can form the incremental investment, and
calculate $\frac{\Delta B}{\Delta C} = \frac{PWB_2 - PWB_1}{PWC_2 - PWC_1} = \frac{AWB_2 - AWB_1}{AWC_2 - AWC_1}$

$$\begin{cases} \text{if } \frac{\Delta B}{\Delta C} > 1 \rightarrow \text{select } I_2 \\ \text{if } \frac{\Delta B}{\Delta C} < 1 \rightarrow \text{Select } I_1 \end{cases}$$

Lecture 25 was revision for midterm

Lecture 26

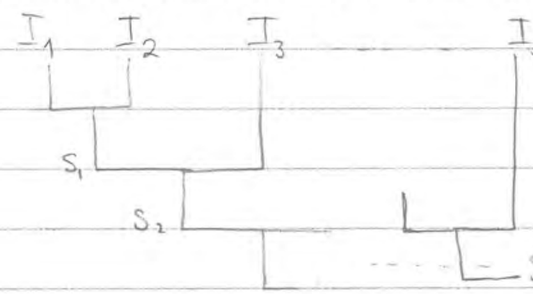
- Applying the Benefit_Cost ratio to the most attractive among "k" exclusive investments. with same analysis period.

	I_1	I_2	I_3	......	I_k
PWB/AWB	..	..	..	..	
PWC/AWC	..	..	..	..	
B/C ratio	$\frac{B_1}{C_1}$	$\frac{B_2}{C_2}$	$\frac{B_3}{C_3}$	—	$\frac{B_k}{C_k}$

Remove the ones with $\frac{B}{C}$ ratio < 1

Suppose there remains "j" investments and suppose they are $I_1, I_2, I_3, \ldots\ldots I_j$ according to increasing PWC or AWC.

Then continue like the old way of Incremental method:



S_{j-1} which is the one you choose, since you already did the B/C for each.

Example.

Consider the 5 mutually exclusive investments.

Year	A	B	C	D	E
0	-200	-100	-125	-150	-225
1-5	68	25	42	52	70

If the MARR is 15% a year, which investment if any is the most attractive.

Start by calculating PW/AW of each

example $PWB_A = 68\,(P/A, 15\%, 5) = 68\,(3.3522) = 228$, $PWC_A = 200$

$B/C_A = 228/200 = 1.14$

	A	B	C	D	E
PWB	228	88	140	174	235
PWC	200	100	125	150	225
B/C	1.14	0.88	1.12	1.16	1.04

↳ < 1 so we remove B

remove B and arrange according to increasing initial cost:

	C	D	A	E
PWB	140	174	228	235
PWC	125	150	200	225

- Now consider the incremental investment (D - C)

$\frac{\Delta B}{\Delta C} = \frac{174 - 140}{150 - 125} > 1 \rightarrow$ Select D

- Now consider the incremental investment (A - D)

$\frac{\Delta B}{\Delta C} = \frac{228 - 174}{200 - 150} > 1 \rightarrow$ Select A.

- Now consider the incremental investment (E - A)

$\frac{\Delta B}{\Delta C} = \frac{235 - 228}{225 - 200} < 1 \rightarrow$ Select A.

Therefore, the most attractive is A with a $B/C = 1.14$.

Example.

A construction is considering different sites to build a dam for a given river.

The Dam's life is assumed to be infinite. The MARR is 6% a year.

Here is the information about the sites.

Site	Construction Cost	Annual Benefits
A	\$6 million	\$350,000
B	\$8 million	\$420,000
C	\$3 million	\$125,000
D	\$10 million	\$400,000
E	\$5 million	\$350,000
F	\$11 million	\$700,000

Based on the Benefit-Cost ratio, which site is more attractive?

infinite life: $A_1 = A_2 = \ldots A$

$P = \frac{A}{i}$

$PWB = A/i$

(in millions)	A	B	C	D	E	F
PWB	5.833	7	2.08	6.66	5.833	11.66
PWC	\$6	8	3	10	5	11
B/C	<1	<1	<1	<1	>1	>1

We eliminate all sites with $B/C < 1$ and we are left with E, F.

We form the incremental (F − E)

$\frac{\Delta B}{\Delta C} = \frac{11.666 - 5.833}{11 - 5} < 1 \longrightarrow$ select E

Therefore, the most attractive is E with B/C ratio of 1.1666

Example. Consider the two investments.

	I_1	I_2
Initial Cost	\$5000	\$6000
Annual Benefits	\$3000	\$3200
Annual Costs	\$300	\$325
Useful Life	3 yrs	6 yrs

If the MARR is 10% a year, which investment is more attractive based on the Benefit-Cost-Ratio.

We first need the same analysis period so we assume $LCM(3,6) = \begin{matrix} 3: 1\times 3 \\ 6: 3\times 2 \end{matrix} = 1 \times 3 \times 2 = 6$

However, we can also use AW and compute over one useful life without the need to extend the cash flow.

	I_1	I_2
AWB	3000	3200
AWC	2310	1703
B/C ratio	1.3	1.88

We form the incremental $(I_1 - I_2)$

$\frac{\Delta B}{\Delta C} < 1$

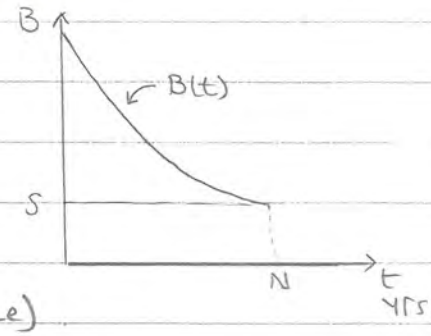
$\therefore$ Select I_2.

Depreciation:

Depreciation is the gradual loss of value of an asset in time.

How to account for depreciation?

One can constrcct a curve given the book value of the asset $B(t)$ such that the curve decreases in time and



$B(0) = B$ (Base Value)

$B(N) = S$ (expected Salvage Value)

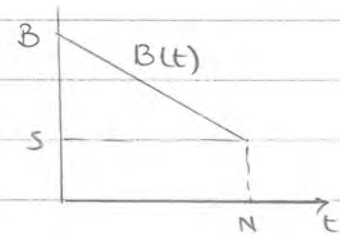
The amount of the depreciation of t_1 and t_2 in $B(t_1) - B(t_2)$

We will use the straight line depreciation method

$$B(t) = B - \frac{(B-S)}{N} t$$

The amount depreciated in a single year is:

$$B(t) - B(t+1) = \frac{B-S}{N}$$



Example- Consider the following electric motor data:

Base Value $10,000

Salvage Value $2,000

depreciable life 5yrs.

Compute the annual depreciation changes and the book values using the SL method (straight line)

Sol.

Year	Annual depreciation	Cummulative dp.
1	$1600	$8400
2	$1600	$6800
3	$1600	$5200
4	$1600	$3600
5	$1600	$2000

$\frac{B-S}{N}$ = annual depreciation = $\frac{10,000 - 2000}{5}$ = 1,600

B → Base Value
S → Salvage Value
N → depreciable life

Uses of depreciation:

* For the evaluation of the assets of a firm. (depreciation for accounting purposes).

* Depreciation for taxation purposes.

By: Ghena Hammour

After tax Economic Analysis.

All governments and states collect taxes from individuals, firms, factories, corporations...

There are many forms of taxes:
Income taxes, sales tax, the added Value tax (VAT / TVA), property tax, input tax...

Different laws determine the amount of taxes.

Some technical terms

* Taxable income: (TI)

TI = Gross Income − Expenditures − deprec. Charges

* Effective tax rate: (TE)

or Incremental tax bracket rate

Taxes = $(TI)(TE)$

Example -

During 4 years, a firm had the following results in $ million

	Year 1	Year 2	Year 3	Year 4
Gross Income	200	150	250	235
purchased machin.	$\boxed{100}$	0	0	0
Expenditures	75	120	120	135

Suppose that the machinery is depriciable by the SL method over 4 years. with zero salvage value.

Compute the annual taxable income.

Note that the $100 million this shall be distributed over 4 years.

$$\text{Annual deprec.} = \frac{100 - \overset{\text{salvage}}{0}}{4} = \$25 \text{ per year.}$$

Now let's calculate the taxable income for each yr.

Year 1 -

$TI = 200 - 25 - 75 = 100\,\$$

Year 2 -

$TI = 150 - 25 - 120 = 5\,\$$

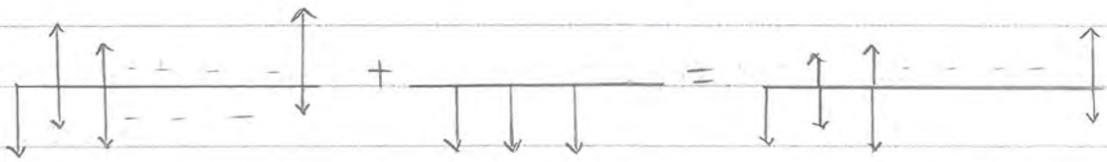
Year 3

$TI = 250 - 25 - 120 = 105\,\$$

Year 4

$TI = 235 - 25 - 135 = 75\,\$$

After tax Economic Analysis.



BTCF (Before Tax Cash Flow) + TCF (Tax Cash Flow) = ATCF (After Tax Cash Flow)

Here use an after tax MARR

Example

A firm invests \$100,000 in new equipment that falls into 3 yrs MACRS category (This equipment is depreciable by the MACRS system).

The projects annual benefits are:

Year	Annual Benefit
1	\$40,000
2	\$50,000
3	\$40,000
4	\$45,000

The effective tax rate is 35% and the after tax MARR is 8%.

a) Find the After tax PW.

b) Calculate the Before and After tax ROR.

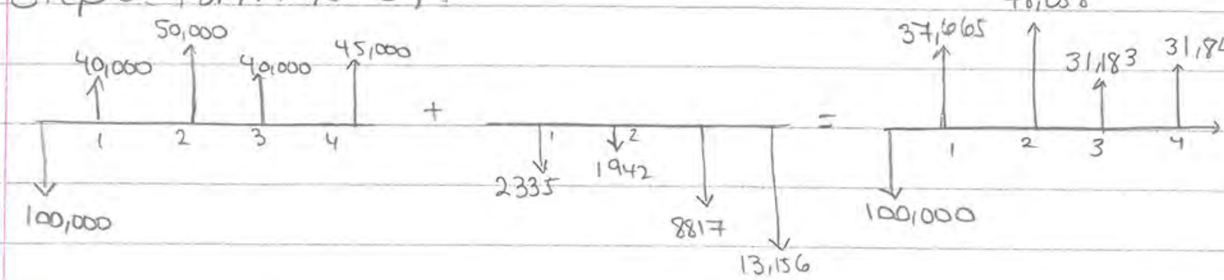
We usually use the SL method, but here just as an example we will use the MACRS.

- In MACRS first 3 years 33% deprec.

Step 1 Form the depreciation and tax schedule.

Year	Annual Benefit	T I = Annual Ben - Expen - dep.	Tax (35%)
1	33,330	40,000 - 33,330 = 6670	2335
2	44,450	50,000 - 44,450 = 5,550	1942
3	14,810	40,000 - 14810 = 25,190	8817
4	7,410	45,000 - 7410 = 37,590	13,156.

Step 2 Form ATCF.



a) After tax PW

PW = -100,000 + 37,765 (P/F, 8%, 1) + 48,058 (P/F, 8%, 2) + 31,183 (P/F, 8%, 3) + 31844 (P/F, 8%, 4)

PW = -100,000 + 37,765 (0.9259) + 48,058 (0.8573) + 31,183 (0.7938) + 31844 (0.7350)

diff ? answer PW = 24325 e

PW = 15828 his answer

b) Before Tax ROR:

PW(i) = -100,000 + 40,000 (P/F, i, 1) + 50,000 (P/F, i, 2) + 40,000 (P/F, i, 3) + 45,000 (P/F, i, 4)

pw(i) = 0

Solving using calc. we get ROR = 26.612%

After tax ROR.

PW(i) = -100,000 + 37665 (P/F, i, 1) + 48,058 (P/F, i, 2) + 31183 (P/F, i, 3) + 31844 (P/F, i, 4)

pw(i) = 0

Solving using calc: we get ROR = 18.98 ~ 19%.

Lecture 29

Example.

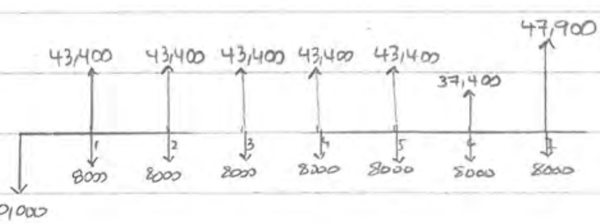
An investment of \$200,000 is good for 7 yrs.
Part of the investment is equipment which amounts of \$100,000 is depreciable by 20% a year for 5 yrs.
The annual costs are \$8,000 and the equip. is expected to be salvaged at \$15,000 after 7 yrs.
The annual benefits are \$50,000.
The effective tax rate is 30%. and the after tax MARR is 8%.
Is the investment attractive based on Present Worth?

Annual dep $= \frac{(100{,}000 - 0)}{5} = 20{,}000$ $\qquad$ TI = Gross Income − Expenditures − deprec.

table.

End of Year	Annual dep.	Annual cost	Annual Benefits	TI	Taxes 30%
1	20,000	8000	50,000	22,000	6,600
2	20,000	8000	50,000	22,000	6,600
3	20000	8000	50,000	22,000	6,600
4	20,000	8000	50,000	22,000	6,600
5	20,000	8000	50,000	22,000	6,600
6	0	8000	50,000	42,000	12,600
7	0	8000	+ Salvage 65,000	57,000	17,100

ATCF − Annual Benefits − Taxes



PW Analysis i = After tax MARR $i = 8\%$ − $PW = 43{,}400(P/A, 8\%, 5) + 37{,}400(P/F, 8\%, 6) + 47{,}900(P/F, 8\%, 7) - 200{,}000 - 8000(P/A, 8\%, 7)$

$PW < 0$

∴ Investment is <u>not</u> attractive!

Example.

A company has two alternative investments.

	A	B
Initial Cost	\$100,000	\$125,000
Annual Benefits	\$40,000	\$55,000
Useful life	4 yrs	5yrs.

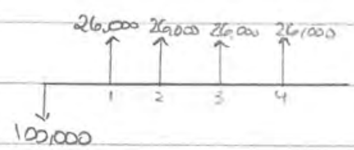
The effective tax rate is 35% and the after tax MARR is 10%.

Based on annual worth, which investment if any is the most attractive.

No analysis period was specified, so we assume LCM, Moreover, Since we will be using AW then we only need to compute over one useful life.

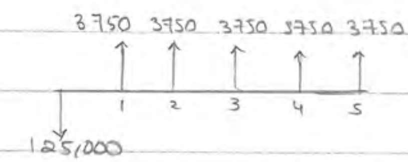
- Form the ATCF for A (ATCF = Benefits - Taxes)

$TI = 40{,}000 - 0$ taxes $= 35\% \times 40{,}000 = 14{,}000$



- Form the ATCF for B

$TI = 55{,}000 - 0$ taxes $= 35\% \times 55{,}000 = 19250$



- AW Analysis (@ $i = 10\%$)

$AW_A = 26000 - 100{,}000(A/P, 10\%, 4) = \$-5547$

$AW_B = 3750 - 125000(A/P, 10\%, 5) = \$2775$

Decision: B is attractive since $AW_B \geq 0$

Example

Consider the two exclusive investments

	A	B
Initial Cost	\$100,000	\$110,000
Annual Benefits	\$30,000	\$35,000
Annual Costs	\$2,000	\$3,000
Useful life	5yrs	5yrs

The after tax MARR is 9% and the effective tax rate is 25%.

Which investment if any is the most attractive?

Use the ROR method.

We first need to form the ATCF for each investment:

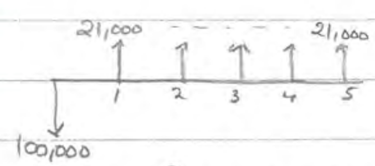
Investment A

$TI = 30{,}000 - 2{,}000 = 28{,}000$ $\quad T = 25\% \times 28{,}000 = 7000\$$

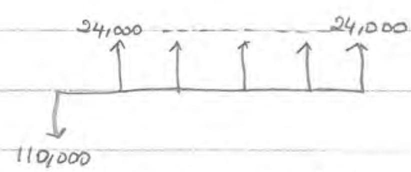
Investment B

$TI = 35{,}000 - 3000 = 32{,}000$ $\quad T = 25\% \times 33{,}000 = 8000\$$

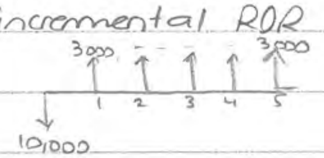
ATCF for A



ATCF B



→ Let us form the incremental ROR then decide (Since same T)



Simple → Unique ROR

$PW(i) = -10{,}000 + 3000\,(P/A, i, 5) = 0$

$15\% < \Delta ROR < 16\%$ → Select B

Now we check ROR of B

$PW(i) = -110{,}000 + 24{,}000\,(P/A, i, 5) = 0$

$ROR_B = 3\% < 9\%$ (After Tax MARR)

∴ Non of the investments are attractive!

Economic Analysis with Inflation

What is inflation?

The loss of purchasing power of money in time.
Future money becomes less valuable than today's money.

Example:

1 \$ in 1975 ≡ \$20 now

Quantitative description and technical terms:

Average annual inflation "f"

It measures the increase in the amount of money needed to buy the same amount of goods and services.

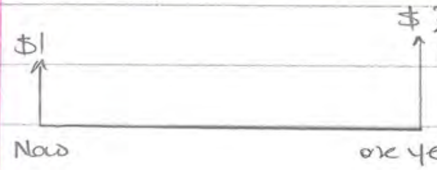
The real rate of growth "i'"

This rate measures the growth of an investment without the effect of inflation.

Inflation adjusted growth rate "i_f" (corporate/market rate)

It measures the growth of an investment taking inflation into account.

Relationship between the 3 rates



$$1(1+i')(1+f) = 1(1+i_f) \Rightarrow (1+i')(1+f) = (1+i_f)$$

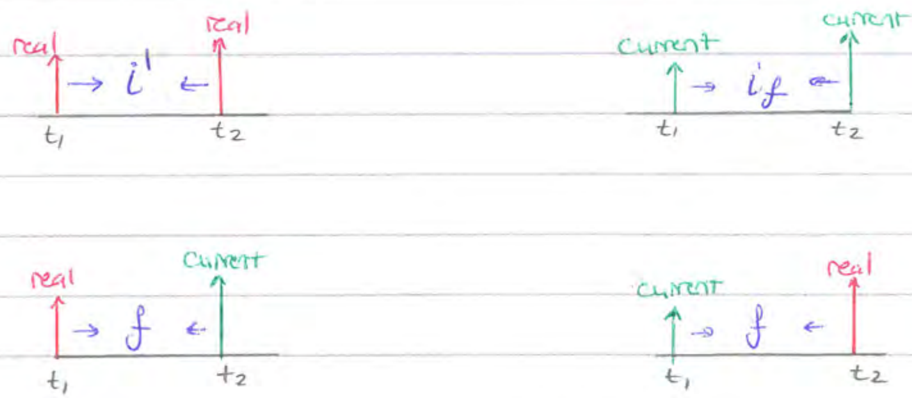
Actual or Current money:

The inflated money or the current money of the year.

Constant or fixed money:

It is the money with purchasing power relative to a fixed year called the base year.

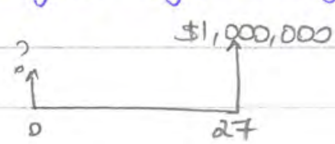
Conversion of money in time:



Example.

A cash value life insurance policy will pay \$1,000,000 when the insured reaches 65. If the insured reaches 65 in 27 yrs from now and the annual average inflation is 3%. What will be the value of the \$1,000,000 in terms of of the purchasing power of today's money?

$f = 3\%$



$P = 1{,}000{,}000\,(P/F,\ 3\%,\ 27)$

$P = \dfrac{1000{,}000}{(1+3\%)^{27}} = \$450{,}187$

Example.

10 years ago, \$1000 was invested at the corporate rate of 9% a year.

The average annual inflation was fixed at 3%.

a) What is real growth rate of the investment?

we have corporate rate $i_f = 9\%$

and the annual avg inf. $f = 3\%$

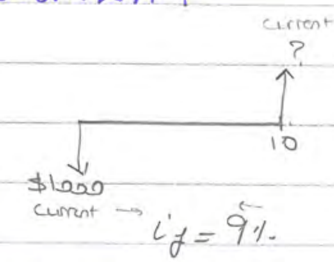
we want real growth rate $i' = ?$

$(1 + i')(1 + f) = (1 + i_f)$

$(1 + i')(1 + 3\%) = (1 + 9\%)$

$i' = 5.8\%$

b) What is the current size of the investment?

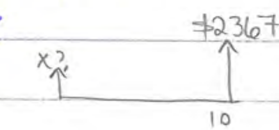


$F = 10{,}000\,(F/P, 9\%, 10)$

$F = \$2367$

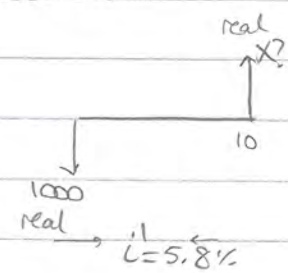
c) What is the size of the investment in terms of the purchasing power of the money 10 years ago?

$X = 2367\,(P/F, 3\%, 10) = \dfrac{2367}{(1+3\%)^{10}} = \$1761$



Or, another way which leads to the same answer:

$X = 1000\,(F/P, 5.8\%, 10) = \$1761$



Economic Analysis with Inflation

Current money of the year
(use corporate MARR)

Actual
Constant money relative to a fixed year
(use real MARR)

$$(1 + \text{corporate MARR}) = (1 + f)(1 + \text{real MARR})$$

Example.

An engineer is to select between two machines "A" an "B"
Information about machine A is in current money.
Information about machine B is in constant money (today's)

	A	B
Initial Cost	\$60,000	\$95,000
Annual Cost	\$55,000	\$35,000
Useful life	10yrs	10yrs.

The company has a real MARR of 15%.
Inflation is expected to be at 5% a year.
Which machine shall be selected based on present worth analysis?

Remember
With current money we use corporate MARR = i_f
With constant/today's money we use real MARR = i'
We have A in current money and B in constant money
and we are given the real MARR = 15% = i', $f = 5\%$
We need to find the corporate MARR i_f to use it with A.

$(1 + \text{Real MARR})(1+f) = (1 + \text{Corporate MARR})$

$(1 + 15\%)(1 + 5\%) = (1 + i_f)$

$i_f = 20.75\%$

Now we proceed normally to perform PW analysis using appropriate rates with each.

$PWC_A = +60{,}000 + 55{,}000(P/A, 20.75\%, 10) = \$\,284{,}837$

$PWC_B = 95{,}000 + 35{,}000(P/A, 15\%, 10) = \$\,270{,}656$

(Remember _ Service investment > minimize cost)

Hence, we select machine B.

Example.

A \$9,000 investment is good for 6 years.
The annual benefits are \$2,300 in terms of today's money.
The investment is depreciable by the SL method for 6 years with zero salvage value.
The effective tax rate is 25%, and the annual inflation rate is 5%.
The after tax corporate rate MARR is 15%.
Is the investment attractive based on present worth?

Annual benefits = 2,300 in terms of today's (constant) money.

Annual inflation rate $f = 5\%$

After tax corporate rate $i_f = 15\%$

Remember:

with constant (today's) money use real MARR = i'.

$(1+i')(1+f) = (1+i_f)$

$(1+i')(1+5\%) = (1+15\%)$

$i' = 9.5238\%$

Annual depreciation = $\frac{9000 - 0}{6} = \$1,500.$

Remember: current → real: $\uparrow \rightarrow f \leftarrow \uparrow$ real → current

depreciation charges are in current money. So we need to convert our constant annual benefits to current benefits using inflation rate $f = 5\%$

Year	Current Benefits
1	$F = (1+5\%)^1(2300) = \$2415$
2	$F = (1+5\%)^2(2,300) = \$2535$
3	$F = (1+5\%)^3(2,300) = \$2663$
4	$F = (1+5\%)^4(2,300) = \$2796$
5	$F = (1+5\%)^5(2,300) = \$2935$
6	$F = (1+5\%)^6(2,300) = \$3082.$

Form the depreciation and tax schedule (so that next we can form the ATCF)

Annual depreciation = 1500

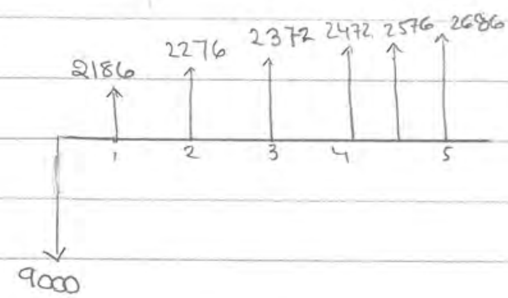
Annual Benefits = as stated in the page before.

TI = GI − D

Year	Annual deprec.	TI	Taxes (25%)
1	1,500	2415−1500 = 915	\$ 229
2	1,500	2535−1500 = 1035	\$ 259
3	1,500	2663−1500 = 1163	\$ 291
4	1,500	2796−1500 = 1296	\$ 324
5	1,500	2935−1500 = 1435	\$ 359
6	1,500	3082−1500 = 1582	\$ 396

Now form the After tax Cash flow: (Annual Benefits − Taxes)

We use After tax corporate MARR since this is current money.



$PW = 2186(P/F, 15\%, 1)$
$+ 2276(P/F, 15\%, 2)$
$+ 2372(P/F, 15\%, 3)$
$+ 2472(P/F, 15\%, 4) + 2576(P/F, 15\%, 5) + 2686(P/F, 15\%, 6)$
$- 9000.$

$PW = \$15.2$

⇒ attractive

Remember

The depreciation charges are in current money and if you are given AW make sure to appropriate conversions.

Example_

Consider two projects A and B.

Financial information about A is in constant money (today's money).

Information about B is in current money of the given year.

The corporate after-tax MARR is 10%.

The inflation rate is 3% a year. $f = 3\%$

The effective tax rate is 25%.

Which is more attractive based on the Benefit_Cost_Ratio.

	A	B
Initial Cost	\$100,000	\$120,000
Annual Cost	\$5000	\$7000
Annual Benefits	\$30,000	\$40,000
Useful life	10 yrs	10 yrs.

- We need to form the ATCF for each.
- For "A" it is in constant money we need to use Real MARR:

$$(1 + \text{Real MARR})(1 + f) = (1 + \text{corporate MARR})$$

$$(1 + \text{Real MARR})(1 + 0.03) = (1 + 0.1)$$

$\Rightarrow$ Real after-tax MARR = 6.796 $\simeq$ 6.8%.

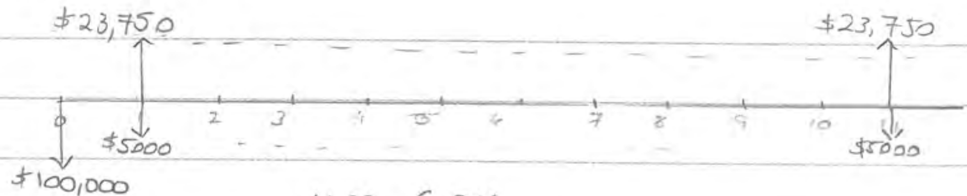
- For "B" it is in current money, we need to use Corporate after tax MARR = 10%.
- Now we can proceed to form the ATCF for each, noting that here we don't have depreciation mentioned.

* ATCF for A.

Calculate $TI = 30{,}000 - 5000 = 25{,}000$ Taxes $= 25\% \times 25{,}000$

Taxes $= 6250$ \$

(Rember ATCF Benefits - Taxes)



use 6.8%

$PWB = 23{,}750\,(P/A, 6.8\%, 11) = \$179881.27$ (Ans. of teacher 162,292)

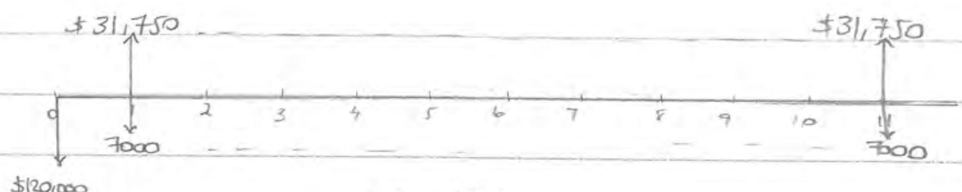
$PWC = +100{,}000 + 5000\,(P/A, 6.8\%, 11) = 137{,}869$ (" " 135,430)

$B/C = \dfrac{179{,}881}{137{,}869} = 1.304 > 1$ (" " 1.24)

* ATCF for B.

Calculate $TI = 40{,}000 - 7000 = 33{,}000$ Taxes $= 25\% \times 33{,}000$

Taxes $= \$8250$



use 10%

$PWB = 31{,}750\,(P/A, 10\%, 11) = 206{,}218$ \$ (Ans. of teacher 195,091)

$PWC = +120{,}000 + 7000\,(P/A, 10\%, 11) = 165{,}465$ \$ (" " 163,012)

$B/C = \dfrac{206{,}218}{165{,}465} = 1.246 > 1$ (" " 1.196)

Now we use the incremental method since both ratios > 1

(B - A)

$\dfrac{\Delta B}{\Delta C} = \dfrac{(206{,}218 - 179{,}881)}{(165{,}465 - 137{,}869)} = 0.954 < 1$ (Ans of teacher > 1)

$\Rightarrow$ Select A. (Ans of teach → Select B)

By: Ghena Hammour

Replacement Analysis.

Equipment, machinery, tools and technical facilities in a firm or company or factory don't serve forever.
It becomes costly to maintain after a certain time.
Therefore, replacement of such things becomes necessary.
However, unplanned replacement can be very costly or catastrophic for example in case of an engine of an airplane!

Replacement Analysis:
Is a collection of Economic techniques that compare the exisisting asset to its potential replacement and help in making the replacement decision.

Technical terms:

Existing asset ----- Defender

Potential replacement ---- Challenger

Type of computations needed to conduct a replacement analysis:

1- Marginal Cost (M_c) for keeping an asset in service:
It is the cost for keeping the asset one more year in service.

M_c = (Operation and Main. Cost) + (Loss in market Value) + (forgone interest).

Example.
A machine has the following information:

Initial Cost	\$50,000
Annual (O and M)	\$3000 for the first two years increases by \$500 every year.
Useful life	7 years.

The estimated Market Values are:

End of Year	Market Value
1	\$35,000
2	\$25,000
3	\$19,000
4	\$16,000
5	\$14,000
6	\$13,000
7	\$12,500

Use a 10% discount rate per year to calculate the marginal costs for keeping the machine 7 years in service.

To calculate the Marginal cost at each year we need:

Mc = (O and M) + (Loss in Market Value) + (Forgone interest)

* Let us do a sample calculation for the first Year:

(O and M) = $3000

Loss in Market Value = 50,000 − 35,000 = 15,000

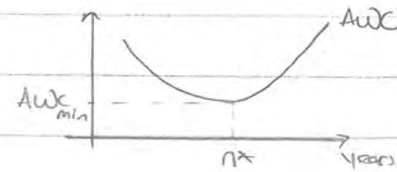
Forgone interest = 50,000 x 10% = 5000

Mc = 3000 + 15,000 + 5000 = 23,000

End of Year	Loss in Market Value	Forgone Interest	(O and M)	Mc
1	50,000 − 35,000 = 15,000	10% x 50,000 = 5000	3000	23,000
2	35,000 − 25,000 = 10,000	10% x 35,000 = 3500	3000	16,500
3	25,000 − 19,000 = 6000	10% x 25,000 = 2500	3500	12,000
4	19,000 − 16000 = 3000	10% x 19,000 = 1900	4000	8,900
5	16,000 − 14000 = 2000	10% x 16,000 = 1600	4500	8,100
6	14,000 − 13,000 = 1000	10% x 14000 = 1400	5000	7,400
7	13,000 − 12,500 = 500	10% x 13,000 = 1300	5500	7,300

2- The economic service life of an asset and the corresponding AWC_{min}:

Is the value of time (n^*) in years at which the AWC attains its minimum value.

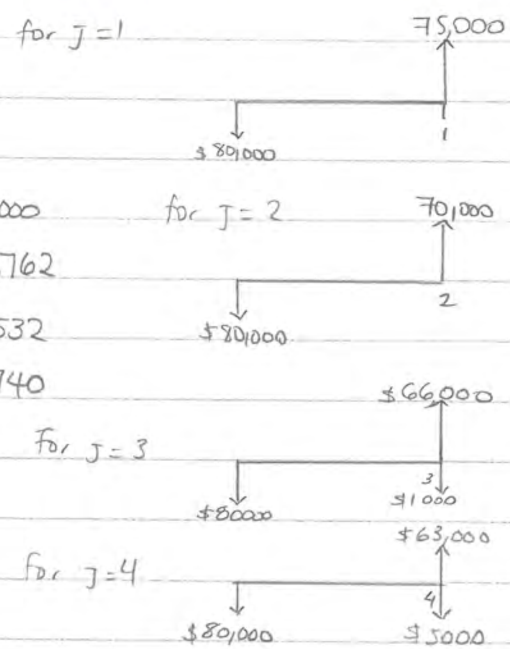


Example.

An asset costs \$80,000 with a useful life 4 years. The estimated salvage values and (O and M) costs are:

Year	Salvage Value	(O and M)
1	\$75,000	0
2	\$70,000	0
3	\$66,000	1,000
4	\$63,000	5,000

Use a 10% discount-rate to calculate the economic service life and the corresponding AWC_{min}.



Year	AWC
1	$80{,}000(A/P, 10\%, 1) - 75{,}000(A/F, 10\%, 1) = 13{,}000$
2	$80{,}000(A/P, 10\%, 2) - 70{,}000(A/F, 10\%, 2) = 12{,}762$
3	$80{,}000(A/P, 10\%, 3) - 65{,}000(A/F, 10\%, 3) = 12{,}532$
4	$80{,}000(A/P, 10\%, 4) - 58{,}000(A/F, 10\%, 4) = 12{,}740$

if we compare we see that

$AWC_{min} = \$12{,}532$

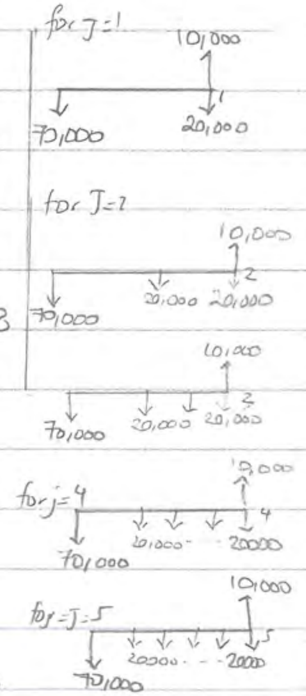
and that:

$n^* = 3$ years.

Example.

A machine costs \$70,000, and its annual cost is 20,000 with an annual salvage value of \$10,000 any time after usage. Use a discount rate of 10% to calculate its economic service life and AWC_{min}. Its useful life is 5 years.

Year	AWC
1	70,000(A/P,10%,1) + 20,000 − 10,000(A/F,10%,1) = 57,000
2	70,000(A/P,10%,2) + 20,000 − 10,000(A/F,10%,2) = 55,571
3	70,000(A/P,10%,3) + 20,000 − 10,000(A/F,10%,3) = 45,126
4	70,000(A/P,10%,4) + 20,000 − 10,000(A/F,10%,4) = 39,928
5	70,000(A/P,10%,5) + 20,000 − 10,000(A/F,10%,5) = 36,828



$AWC_{min} = 36,828\ \$$

$n^* = 5$

Replacement Technique _1_

Information Available:

Defender	Challenger
Marginal costs that are increasing.	n^*
	AWC_{min}

We compare the marginal costs of the defender to the AWC_{min} of the Challenger.

We replace at the beginning of the year at which:

$$M_C \text{ (at the given yr)} > AWC_{min}$$

Example.

Information on a defender and its challenger given are:

Defender Year	MC
1	\$13,250
2	\$14,600
3	\$15,750
4	\$17,300
5	\$18,650

Challenger

$n^* = 4$ years

$AWC_{min} = \$15,430$

What is the replacement decision?

We first realize that we are given Defender with MC increasing

So we use Technique_1_ and start comparing each MC with AWC_{min} until it's greater than the AWC_{min}.

@ Year 3 $MC = 15750 > AWC_{min} = 15,430$

Decision: Keep the defender two more years then replace.

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Replacement Technique -2-

Information Available.

Defender	Challenger
Marginal Costs	n^*
but aren't increasing.	AWCmin.

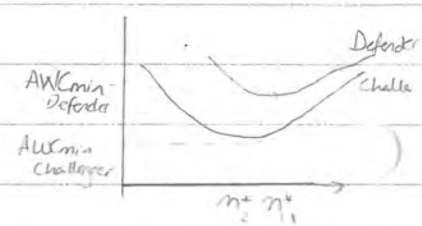
We compute AWCmin for the defender over its remaining years in service, then;

Case 1

AWCmin (defender) > AWCmin (challenger)

Decision:

Replace Now

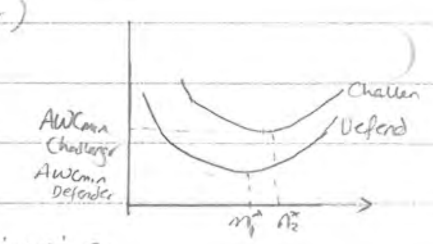


Case 2

AWCmin (Defender) < AWCmin (Challenger)

Decision:

keep the defender at least n_1^* years in service.



Replace after n_1^* years at the beginning of the year at which the Marginal Cost of the defender is higher than the AWCmin of the Challenger.

(MC (defender at that year) > AWCmin (challenger))

Example

The market value of a machine in use now is \$5000.
Its remaining years of service are 4 yrs.
The annual costs are \$1000, Its salvage Values are:

End of Year	Salvage Value
1	\$2000
2	\$1000
3	\$500
4	\$200

This machine can be replaced by another one with a cost of \$10,000, annual cost of \$800, useful life of 6 years, and estimated salvage values are:

End of Year	Salvage Value
1	\$8,600
2	\$6,000
3	\$4,000
4	\$2,000
5	0
6	0

At a 10% discount rate a year, what is the replacement decision?

So, we start by computing the MC of the Defender then see if they are increasing then we proceed to use replacement technique 1, if they are not increasing we use Replacement technique 2.

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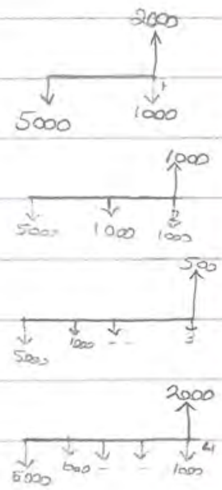
* Compute Marginal Cost of defender.

MC = (O and M) + (Loss in market Value) + (Forgone interest)

End of Year	Loss in Market Value	Forgone interest	(M and O)	MC
1	5000 - 2000 = 3000	10% x 5000 = 500	1000	4,500
2	2000 - 1000 = 1000	10% x 2000 = 200	1000	2,200
3	1000 - 500 = 500	10% x 1000 = 100	1000	1,600
4	500 - 200 = 300	10% x 500 = 50	1000	1,350

* MC are not increasing so we use technique 2 and Compute AWCmin of the defender.

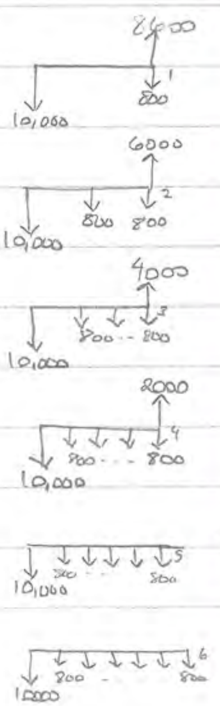


J	AWC
1	5000 (A/P, 10%, 1) - 2000 (A/F, 10%, 1) + 1000 = 4500
2	5000 (A/P, 10%, 2) - 1000 (A/F, 10%, 2) + 1000 = 3405
3	5000 (A/P, 10%, 3) - 500 (A/F, 10%, 3) + 1000 = 2860
4	5000 (A/P, 10%, 4) - 200 (A/F, 10%, 4) + 1000 = 2536

We note that:

$n_1^* = 4$ yrs , AWCmin = \$2536.

* Now we compute AWCmin of Challenger to compare:



J	AWC
1	10,000 (A/P, 10%, 1) - 8600 (A/F, 10%, 1) + 800 = 3500
2	10,000 (A/P, 10%, 2) - 6000 (A/F, 10%, 2) + 800 = 3705
3	10,000 (A/P, 10%, 3) - 4000 (A/F, 10%, 3) + 800 = 3540
4	10,000 (A/P, 10%, 4) - 2000 (A/F, 10%, 4) + 800 = 3524
5	10,000 (A/P, 10%, 5) + 800 = 3438
6	10,000 (A/P, 10%, 6) + 800 = 3096

$n^* = 6$, AWCmin = \$3096.

* We compare AWCmin (defender) = \$2536 < AWCmin (Challenger) = \$3096

Decision: Keep defender at least 4 yrs (which is the useful life in this case) then replace.

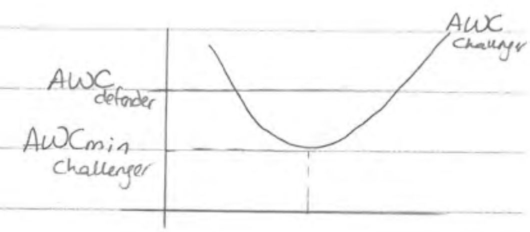
Replacement Technique -3-

Information Available:

Defender	Challenger
Marginal Costs are not available.	n^* AWC_{min}

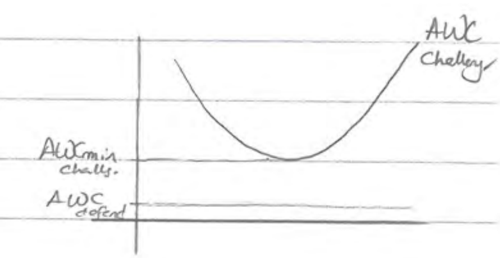
Case-1-

Decision:
Replace Now



Case-2-

Decision:
Keep the defender over its remaining years in service.



Example:

An existing machine has an estimated value of \$1200 and 3 more years in service.
The annual cost is \$800 and the salvage value is 0 after 3 years.
It's replacement has the following information:

Initial Cost \$ 5000
Annual Cost \$100
Salvage Value 0 after any year.

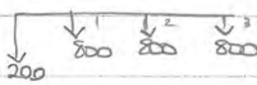
At a 10% discount rate, what is your replacement decision?

From the information given, the marginal costs for the defender cannot be computed → Use Technique 3.

First,

Compute AWC of defender

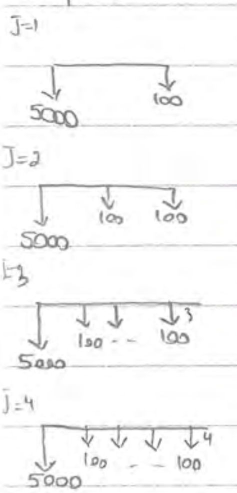
$AWC = 1200(A/P, 10\%, 3) + 800 = \$\underline{\underline{1283}}$



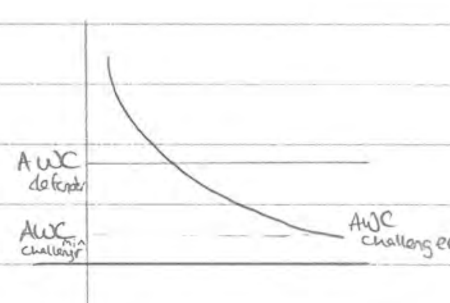
Second,

Compute AWC_{min} for challenger:

remember to Compute AWC_{min}



J	AWC
1	$5000(A/P, 10\%, 1) + 100 = 5600$
2	$5000(A/P, 10\%, 2) + 100 = 2981$
3	$5000(A/P, 10\%, 3) + 100 = 2111$
4	$5000(A/P, 10\% 4) + 100 = 1678$
5	$5000(A/P, 10\%, 5) + 100 = 1319$
6	$5000 + (A/P, 10\% 6) + 100 = 1245$
7	$5000(A/P, 10\%, 7) + 100 = \underline{\underline{1127}}$



$\therefore AWC_{defender} > AWC_{min\ challenger}$

⇒ Decision: Replace Now!

Side Note:

The difference between mutually exclusive and independent investments is:

For a mutually exclusive ones, we can select at most one investment.

For independent investments we can select as much as the budget allows.

Selection from independent projects under budget limitations:

Given "k" projects with initial costs: $I_1, I_2, \ldots I_k$

The available budget is "b".

What project(s) should be selected that maximize the profit while meeting the budget contstraint.

1. Identify the feasible collections of projects which satify the budget constraint.
   (Sum of initial costs $\leq b$)

2. Find the PW of each feasible collection:
   Sum of the present worth of all projects in the collection.

3. Select the feasible collection with the highest PW.

Example

Given the four independent projects:

	A	B	C	D
Initial Cost	\$50,000	\$40,000	\$35,000	\$25,000
Annual Benefit	\$40,000	\$25,000	\$20,000	\$17,000
Useful life	6yrs	4yrs	6yrs	4yrs

The MARR is 10% and the available budget is \$100,000

which projects should be selected?

Let us start by finding PW of each project.

project	A	B	C	D
PW	\$124,212	\$39,250	\$52106	\$28,890

- Collections of 4 is not feasible since,
  $\sum$ Initial costs = 150,000\$ > b = 100,000\$
- Collections of 3

Collection	Feasibality	PW feasability
(A,B,C)	×	/
(A,B,D)	×	/
(A,C,D)	×	/
(B,C,D)	✓ since: 62,000	× since: 120,246.

- Collections of 2.

we proceed to collections of 2 since although the sum of there initial costs is less than the budget but the sum of their PW is not!

$\frac{4!}{2!\,2!} = 6$

Collection	feasability	PW feasability
(A,B)	✓	163,462 X
(A,C)	✓	176,318 X
(A,D)	✓	153,102 X
(B,C)	✓	91,356
(B,D)	✓	68,140
(C,D)	✓	80,996

we select the one with the highest PW,
Hence, select (B,C).

By: Ghena Hammour

Solving the capital budgetting problem using Integer Linear Programming (ILP)

ILP is an optimization method that can be used to solve the problem quickly using a computer.

The formulation is as follows:

	Project 1	project 2	---	Project k
Initial Cost	I_1	I_2	---	I_k
PW	PW_1	PW_2	---	PW_k

budget b

Steps are:

- maximize $z = \sum_{j=1}^{k} x_j PW_j$
- Subject to the constraint: $\sum_{j=1}^{k} x_j I_j < b$

where

$x_j = \begin{cases} 0 \\ 1 \end{cases}$

- The output of this algorithm:

output: $(x_1^*, x_2^*, x_3^*, \ldots, x_k^*)$

↳ if it is 0 then that project isn't selected
if it is 1 then the project is selected.

Introduction to Economic Analysis under risk.

If at least one of the parameters of an investment like; initial cost, annual cost, annual benefit... is a random varaible with a know probability distribution, the investment is called under risk.

How to assess the economic worth of a random investment?

1. The Expected present worth Criterion:
   $E(PW) > 0$ ⟶ attractive
   $E(PW) < 0$ ⟶ not attractive

2. The negative present worth probability Ceiling:
   If $P(PW<0) > P_0$ — reject.

Side discussion

X

Discrete Type random Variable

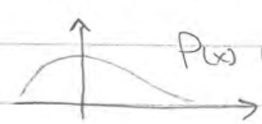
$X = x_1, x_2 \ldots$

$p(X = x_i) = p_i$

↳ probability mass fnct

$\sum p_i = 1$

Continuous Type. random variable.



$P(x)$ probality density fnct.

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

How to find the expected value:

$$E(x) = \begin{cases} \sum x\,p(x) & \text{if } x \text{ is discrete} \\ \int_{-\infty}^{\infty} x\,p(x)\,dx & \text{if } x \text{ is continuous} \end{cases}$$

Example

An investment has the following possible cash flows with the corresponding probabilities.

Year	Receding $P_r = 0.2$	Stable $P_s = 0.6$	Expanding $P_r = 0.2$
1	2,500	2000	2000
2	2,000	2000	3000
3	1,000	2000	5500

The initial cost is \$5000, and the MARR is 15% a year. Is the investment attractive? Use the expected PW criterion.

$PW\ (\text{receding}) = -5000 + 2500(P/F,15\%,1) + 2000(P/F,15\%,2) + 1000(P/F,15\%,3) = -656$

$PW\ (\text{stable}) = -5000 + 2000(P/A,15\%,3) = -434$

$PW\ (\text{Expanding}) = -5000 + 2000(P/F,15\%,1) + 3000(P/F,15\%,2) + 5500(P/F,15\%,3) = 1309$.

Expected Present Worth

$E(PW) = -656(0.2) - 434(0.6) + 1309(0.2)$

$E(PW) = -130 < 0 \Rightarrow$ Not attractive.

Example.

An investment has the following distributes for PW

PW	-500	-300	-100	100	600	800	1000
Pr	0.25	0.1	0.05	0.1	0.2	0.15	0.15

Is the investment attractive based on the expected PW criterion?

$$E(PW) = -500(0.25) - 300(0.1) - 100(0.05) + 100(0.1) + 600(0.2) + 800(0.15) + 1000(0.15) = 340 > 0$$

$\Rightarrow$ Investment is attractive.

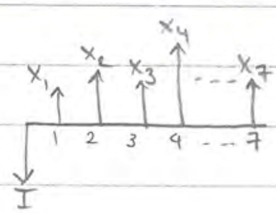
A random investment is good for 7 years. The annual benefit is a random variable x with:

pdf: $f(x) = \begin{cases} 2x & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$, (one unit of $x = \$10,000$)

The MARR is 12% a year. What is the range of values of the initial cost so that the investment is attractive based on the expected PW criterion.

Let the initial cost be I

$$PW = \sum_{j=1}^{7} x_j \, (P|F, 12\%, j) - I$$

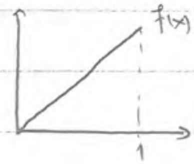


Note that:

$E(ax) = a\,E(x)$

$E(g_1(x) + g_2(x)) = E(g_1(x)) + E(g_2(x))$

$$E(PW) = \sum_{j=1}^{7} x_j (P/F, 12\%, j) - I$$



$$E(x_j) = E(x) = \int_0^1 (2x\,dx) \cdot 10{,}000$$

$$= \frac{2x^3}{3}\Big]_0^1 (10{,}000)$$

$$= 6667\$$$

$$E(PW) = \left(\sum_{j=1}^{7} (P/F, 12\%, 7)\right)(6667) - I$$

$$E(PW) = 30{,}426 - I$$

The investment is attractive if $E(PW) > 0$

$\Rightarrow \; 30{,}426 - I > 0$

$I < \$\,30{,}426$

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